Lipschitz stability for the electrical impedance tomography problem: the complex case.

Elena Beretta (Università di Roma "La Sapienza")

in collaboration with E. Francini

November 8, 2010

Outline

1. The forward problem
2. The inverse problem
3. The main result
4. Main ingredients of the proof
5. Concluding remarks

Part I

The forward problem

Formulation of the problem

Given a bounded connected open set $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, given γ a bounded complex valued function (admittivity function) satisfying

$$\Re \gamma \geq \lambda^{-1} > 0$$

and given $f \in H^{1/2}(\partial\Omega)$ let us consider the weak solution $u \in H^1(\Omega)$ of the problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega. \end{cases}$$

Consider the Dirichlet to Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}_{|\partial\Omega},$$

where ν is the exterior unit normal vector to $\partial\Omega$.

Part II

The inverse problem

Formulation of the problem

EIT Inverse Problem:

Determine γ from the knowledge of the Dirichlet-to Neumann map Λ_γ

Known results

Uniqueness

- $n \geq 3$ If γ is sufficiently smooth the results obtained by Sylvester and Uhlmann (1987) for the conductivity case can be extended to the complex case.
- $n = 2$ Francini (2000), Bukhgeim (2008).
- $n \geq 3, \gamma \in L^{\infty}(\Omega)$ completely open
- $n = 2$ Astala and Paivarinta for real conductivities $\gamma \in L^{\infty}(\Omega)$

Known results

Stability

- $n \geq 3$ Alessandrini (1988) has proved for the conductivity case that if

$$\|\gamma\|_{W^{2,\infty}(\Omega)} \leq E$$

  then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|)$$

  where $\omega(t) = C|\log t|^{-\eta}$.

- $n = 2$ Barcelo, Faraco and Ruiz (2007) have proved for the conductivity case that if

$$\|\gamma\|_{C^\alpha(\overline{\Omega})} \leq E$$

  then logarithmic continuous dependence holds.

- Clop, Faraco, Ruiz (2009) $\gamma \in W^{\alpha,p}$, $\alpha > 0$ and $1 < p < \infty$ then

$$\|\gamma_1 - \gamma_2\|_{L^2(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|)$$

  where $\omega(t) = C|\log t|^{-\eta}$.

A-priori assumptions

Mandache (2001) has proved that logarithmic stability is the best possible stability also using as a-priori assumption

$$\|\gamma\|_{C^k(\overline{\Omega})} \leq E, \quad \forall k = 0, 1, 2....$$

Strategy:

Look for a-priori assumptions on γ

that are

- physically relevant
- give rise to a better type of stability

A-priori assumptions

Alessandrini and Vessella (2003) assume that

γ piecewise constant

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j 1_{D_j}(x) \quad \text{a.e. in } \Omega,$$

where $D_1, \ldots D_N$ are known disjoint Lipschitz domains and $\gamma_1, \ldots \gamma_N$ are unknown real numbers.

Assuming ellipticity and $C^{1,\alpha}$ regularity at the interfaces joining contiguous domains D_j and at $\partial\Omega$ they prove Lipschitz continuous dependence of γ from Λ_γ

Part III

The main result

Main assumptions

(H1) $\Omega \subset \mathbb{R}^n$ is a bounded domain such that

$$|\Omega| \le A,$$

and

$\partial\Omega$ is of Lipschitz class with constants r_0, L.

(H2) The complex conductivity γ satisfies

$$\Re\gamma \ge \lambda^{-1}, \quad |\gamma| \le \lambda \qquad \text{a.e. in } \Omega$$

for some $\lambda \ge 1$, and is of the form

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j 1_{D_j}(x),$$

where γ_j are for $j = 1, \ldots, N$ unknown complex numbers and D_j are known open sets in $\mathbb{R}^n$ which satisfy the following conditions

Main assumptions

(H3) $D_i \cap D_j = \emptyset$, $i \neq j$ such that $\cup_{j=1}^{N} \overline{D}_j = \overline{\Omega}$, ∂D_j, $j = 1, \ldots, N$ of Lipschitz class with constants r_0, L.

There exists one region D_1 such that $\partial D_1 \cap \partial \Omega$ contains an open flat portion Σ_1. For every $j \in \{2, \ldots, N\}$ there exists a walkway from D_1 to D_j, i.e. there exist $j_1, \ldots, j_M \in \{1, \ldots, N\}$ such that

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \ldots, M$

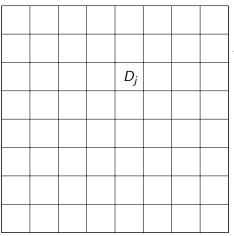$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a flat portion Σ_k such that

$$\Sigma_k \subset \Omega, \quad \forall k = 2, \ldots, M.$$

Main assumptions

There exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates such that $P_k = 0$ and

$$\begin{aligned}
\Sigma_k \cap B_{r_0/3} &= \{x \in B_{r_0/3} : x_n = 0\}, \\
D_{j_k} \cap B_{r_0/3} &= \{x \in B_{r_0/3} : x_n > 0\}, \\
D_{j_{k-1}} \cap B_{r_0/3} &= \{x \in B_{r_0/3} : x_n < 0\}.
\end{aligned}$$

Main assumptions

A possible configuration

$$\overline{\Omega} = \cup_{j=1}^{N} \overline{D}_j$$

Main result

Theorem

Let Ω satisfy assumption **(H1)**. Let $\gamma^{(k)}$, $k = 1, 2$ be two complex piecewise constant functions of the form

$$\gamma^{(k)}(x) = \sum_{j=1}^{N} \gamma_j^{(k)} 1_{D_j}(x),$$

where $\gamma^{(k)}$ satisfy for $k = 1, 2$ assumption **(H2)** and $D_j, j = 1, \ldots, N$ satisfy assumption **(H3)**. Then there exists a positive constant $C = C(r_0, L, A, n, N, \lambda)$ such that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda_1 - \Lambda_2\|_{\mathfrak{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}, \tag{1}$$

where $\Lambda_k = \Lambda_{\gamma^{(k)}}$ for $k = 1, 2$.

[B., Francini, 2010]

Part IV

Proof of the main result

Main ideas of the proof

The proof is constructive and is based on:

- Quantitative estimates of unique continuation of harmonic functions
- Construction of singular solutions and study of its asymptotic behaviour near the discontinuity interfaces.
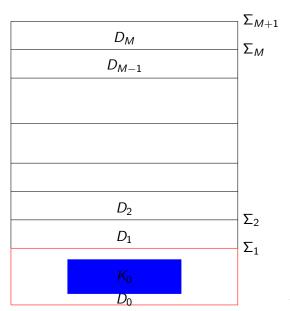
Estimates of unique continuation

We extend Ω to $\Omega_0 = \Omega \cup D_0$ by adding an open cylinder D_0 whose basis is the flat portion Σ_1 of $\partial\Omega \cap \overline{D}_1$ and with height greater than r_0. Let $K_0 = \{x \in D_0 : \text{dist}(x, \partial D_0) \geq r_0/2\}$.

Extend γ on Ω_0 setting it equal to 1 in D_0. Let us consider the chain of domains connecting D_0 to D_M, $M \leq N$ where

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_M)} = \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)}.$$

and consider the *walkway* $K \subset \cup_{j=0}^{M} \overline{D}_j$ with Lipschitz boundary such that $\overline{K} \cap \partial D_j = \Sigma_j \cup \Sigma_{j+1}$ for $j = 1, \ldots, M$ and $\text{dist}(K, \partial(\cup_{j=0}^{M} \overline{D}_j) \setminus \{\Sigma_{M+1} \cup D_0\}) > r_0/16$.

The set K

Estimates of unique continuation

Theorem

Let $v \in H^1(K)$ be a solution to

$$div(\gamma \nabla v) = 0 \quad in \quad K,$$

such that

$$\|v\|_{L^\infty(K_0)} \leq \epsilon_0,$$

$$|v(x)| \leq (\epsilon_0 + E_0) dist(x, \Sigma_{M+1})^{1-\frac{n}{2}} \quad for \; every \quad x \in K.$$

Then

$$|v(\tilde{x})| \leq C \left(\frac{\epsilon_0}{E_0 + \epsilon_0} \right)^{\tau^{(M+1)N_1} \delta_1^{M+1} \tau_r} (E_0 + \epsilon_0) r^{\left(1-\frac{n}{2}\right)(1-\tau_r)},$$

where $\tilde{x} = P_{M+1} - 2r\nu(P_{M+1})$, $r \in (0, r_0)$ and $\tau, \tau_r, \delta_1 \in (0,1)$, $N_1 = \frac{A}{r_0^n}$ and C depends on r_0, L, A, n, λ, N.

Singular solutions

Difficulty:

Lack of a result of existence of the Green's function in Ω for $n \geq 3$

- $n \geq 3$ Hofmann and Kim (2007) existence of Green's function for weak solutions satisfying interior Hoelder continuity estimates .
- $n = 2$ Dong and Kim (2009) existence of Green's function

Singular solutions

Theorem

Let γ satisfy assumptions **(H1)-(H3)** in Ω_0. For $y \in K$ there exists a unique function $G(\cdot, y)$, continuous in $\Omega \setminus \{y\}$

$$\int_{\Omega_0} \gamma \nabla G(\cdot, y) \cdot \nabla \phi = \phi(y), \quad \forall \phi \in C_0^\infty(\Omega_0),$$

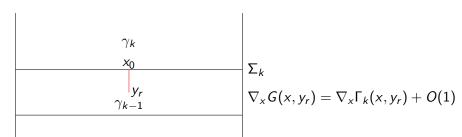$$\|G(\cdot, y)\|_{H^1(\Omega \setminus B_r(y))} \leq C r^{1-n/2}, \quad \forall r < d_y/2,$$

$$G(x, y) = G(y, x) \quad \text{for every } x, y \in K.$$

Moreover for $x \in D_k$, $x_0 \in \Sigma_k$ and $y_r = x_0 + r\nu$,

$$\nabla_x G(x, y_r) = \nabla_x \Gamma_k(x, y_r) + O(1)$$

Singular solutions

$$\operatorname{div}\left(\left(\gamma_{k-1}1_{\mathbb{R}^n_-} + \gamma_k 1_{\mathbb{R}^n_+}\right)\nabla_x\Gamma_k(\cdot, y)\right) = \delta_y$$

γ_k

x_0

Σ_k

y_r

γ_{k-1}

$\nabla_x G(x, y_r) = \nabla_x\Gamma_k(x, y_r) + O(1)$

Proof

Let $D_0, D_1, \ldots, D_M$ be the chain of domains such that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_M)} = E := \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)}.$$

and let

$$\delta_k := \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\cup_{j=0}^k D_j)}$$

for $k = 1, \cdots, M$.

Proof

Note that, for $y, z \in D_0$ we have

$$\int_\Omega (\gamma^{(1)} - \gamma^{(2)}) \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, z) = < (\Lambda_1 - \Lambda_2) G_1(\cdot, y), \overline{G_2(\cdot, z)} >$$

so that for $y, z \in K_0$

$$\left| \int_\Omega (\gamma^{(1)} - \gamma^{(2)}) \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, z) \right| \le C\epsilon$$

Then

$$\left| \int_{\Omega \setminus \cup_{j=1}^{k-1} D_j} (\gamma^{(1)}(x) - \gamma^{(2)}(x)) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, z) \, dx \right| \le C(\epsilon + \delta_{k-1})$$

Define, for $y, z \in K \cap \cup_{j=1}^{k-1} D_j$,

$$S_{k-1}(y, z) := \int_{\Omega \setminus \cup_{j=1}^{k-1} D_j} (\gamma^{(1)}(x) - \gamma^{(2)}(x)) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, z) \, dx$$

Proof

We have

$$|S_{k-1}(y,z)| \leq C(\epsilon + \delta_{k-1}) \quad \forall y, z \in K_0,$$

$$|S_{k-1}(y,z)| \leq CE \left(d(y, \Sigma_k) d(z, \Sigma_k)\right)^{1-\frac{n}{2}} \forall y, z \in K \cap \cup_{j=1}^{k-1} D_j,$$

and

$$\text{div}\left(\gamma^{(1)} \nabla S_{k-1}(\cdot, z)\right) = 0, \quad \text{div}\left(\gamma^{(2)} \nabla S_{k-1}(y, \cdot)\right) = 0 \quad \text{in } K \cap \cup_{j=1}^{k-1} D_j.$$

Using the Hoelder estimates of unique continuation in $K \cap W_{k-1}$ we get

$$|S_{k-1}(y_r, y_r)| \leq C \left(\frac{\epsilon + \delta_{k-1}}{\epsilon + \delta_{k-1} + E}\right)^{\alpha_r} (\epsilon + \delta_{k-1} + E) r^{2-n}$$

Proof

Using the asymptotic behaviour of $\nabla_x G_j$ we have

$$|\gamma_k^{(1)} - \gamma_k^{(2)}| \leq C(r^{n-2}|S_{k-1}(y_r, y_r)| + E r^{n-2}) \quad \forall r < r_0/8$$

from the Hoelder estimates for $S_{k-1}(y_r, y_r)$ we derive

$$|\gamma_k^{(1)} - \gamma_k^{(2)}| \leq C(\epsilon + \delta_{k-1} + E)\left[\left(\frac{\epsilon + \delta_{k-1}}{\epsilon + \delta_{k-1} + E}\right)^{(\tau^{kN_1}\delta_1^k\tau_r)^2} + \left(\frac{r}{r_0}\right)^{n-2}\right]$$

and we get

$$\delta_k \leq C(\epsilon + \delta_{k-1} + E)\omega\left(\frac{\epsilon + \delta_{k-1}}{\epsilon + \delta_{k-1} + E}\right). \tag{2}$$

where

$$\omega(t) = \begin{cases} |\log t|^{-(n-2)/4} & \text{for} \quad 0 < t \leq \frac{1}{e^n} \\ n^{-(n-2)/4} & \text{for} \quad t \geq \frac{1}{e^n} \end{cases}$$

Proof

Since $\delta_0 = 0$ iterating we obtain

$$\delta_k + \epsilon \le (C+1)^k (E+\epsilon)\omega_k \left(\frac{\epsilon}{\epsilon + E} \right),$$

where ω_k is the composition of ω with itself k times.
Now we recall that $E = \delta_M$ and, hence,

$$E + \epsilon \le (C+1)^M (\epsilon + E)\omega_M \left(\frac{\epsilon}{\epsilon + E} \right)$$

Now, either $E \le \epsilon$ and this proves Lipschitz stability, or $E > \epsilon$ and we can write

$$E \le 2(C+1)^M E \omega_M \left(\frac{\epsilon}{2E} \right),$$

from which

$$E \le \frac{\epsilon}{2\omega_M^{-1}(\frac{1}{2(C+1)^M})}$$

Part V

Conclusions

Final remarks

Remark1

We expect that Lipschitz continuous dependence of the admittivities on the data still holds replacing the flatness condition on the interfaces with $C^{1,\alpha}$ regularity.

Final remarks

Remark2

In our stability estimate the constant $C = C(r_0, L, A, n, N, \lambda)$ and it diverges exponentially in the number N of the subdomains of the partition of Ω.

Rondi has proved in (2006) that

$$C \geq A exp(BN^{1/2n})$$

with A and B absolute constants. This inequality can be interpreted as a resolution limit in the determination of γ from Λ_γ . If ϵ is the error on the measured data and we can tolerate an error up to $\overline{C}\epsilon$ ($\overline{C}$ error amplification tolerance) and if all the domains D_j have size r_0^n then $r_0 = N^{-1/n}$ and we get

$$r_0 \geq \left(\frac{1}{B} \log \frac{\overline{C}}{A} \right)^{-(2n-1)/n}$$

So if we fix $\overline{C}$ no detail smaller than $(\frac{1}{B} \log \frac{\overline{C}}{A})^{-(2n-1)/n}$ can be detected.

Final remarks

Remark3

Observe that in our result we can replace the full Dirichlet to Neumann map with the local Dirichlet to Neumann map. More precisely, let Σ be an open portion of $\partial\Omega$ containing a flat open subset. Let $H^{1/2}_{co}(\Sigma) = \{\phi \in H^{1/2}(\partial\Omega) : \operatorname{supp} \phi \subset \Sigma\}$ and define the local Dirichlet to Neumann map in the following way: for $\phi \in H^{1/2}_{co}(\Sigma)$ let

$$< \Lambda^{\Sigma}_{\gamma}\phi, \psi > = \int_{\Omega} \gamma \nabla u \nabla \overline{v} \quad \text{for } \phi, \psi \in H^{1/2}_{co}(\Sigma)$$

where u solves equation $\operatorname{div}(\gamma \nabla u) = 0$ and $u = \phi$ on $\partial\Omega$ and $v \in H^1(\Omega)$ such that $v = \psi$ on $\partial\Omega$.

We observe that in our proof we apply the Dirichlet to Neumann map to functions whose support is contained in a neighborhood of the flat portion of $\partial\Omega$. Hence we have

$$\|\gamma^1 - \gamma^2\|_{L^{\infty}(\Omega)} \leq C\|\Lambda^{\Sigma}_1 - \Lambda^{\Sigma}_2\|_{\mathfrak{L}(H^{1/2}_{co}, H^{-1/2}_{co})}.$$