Inverse problem for parabolic systems

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[1] Joint work with M. Cristofol, P. Gaitan, L. de Teresea and M. Yamamoto

Outline

1. Introduction to the inverse problem
2. Carleman estimates and inverse problem for scalar parabolic operators
3. Previous result
4. The first main result : 2×2 parabolic systems with convection terms
5. The second main result : 3×3 parabolic systems
6. Link with observability of parabolic systems
7. Comments

The problem

- The direct problem

$$\begin{cases} \partial_t Y = D\Delta Y + C \cdot \nabla Y + AY, & \text{in } (0,T)\times\Omega \\ Y = h, & \text{on } (0,T)\times\partial\Omega \\ Y(0) = Y_0, & \text{in } \Omega \end{cases} \tag{1.1}$$

$$A = (a_{ij}) \in L^\infty(\Omega;\ M_n(\mathbb{R})), \quad D = diag(d_i), \quad C \in L^\infty(\Omega, M_n(\mathbb{R}^n))$$

- The observation

$$B \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m) \quad m \le n, \quad \omega \subset\subset \Omega$$

Is it possible to recover $(a_{i,j})$ by the knowledge of BY in $(0,T)\times\omega$?

Identifiability

$$\begin{cases} \partial_t Y_1 = D\Delta Y_1 + C\cdot\nabla Y_1 + A_1 Y_1, & \text{in } (0,T)\times\Omega \\ Y_1 = h, & \text{on } (0,T)\times\partial\Omega \\ Y_1(0) = Y_{10}, & \text{in } \Omega \end{cases} \tag{1.2}$$

$$\begin{cases} \partial_t Y_2 = D\Delta Y_2 + C\cdot\nabla Y_2 + A_2 Y_2, & \text{in } (0,T)\times\Omega \\ Y_2 = h, & \text{on } (0,T)\times\partial\Omega \\ Y_2(0) = Y_{20}, & \text{in } \Omega \end{cases} \tag{1.3}$$

Identifiability

$$U = Y_1 - Y_2$$

$$\left.\begin{cases} \partial_t U = D\Delta U + A_1 U + C \cdot \nabla U + (A_1 - A_2) Y_2, \text{ in } (0,T) \times \Omega \\ \\ U = 0, \text{ on } (0,T) \times \partial\Omega \\ \\ BU = 0, \text{ in } (0,T) \times \omega \end{cases}\right\} \overset{?}{\Rightarrow} A_1 = A_2$$

For example $B = (1, 0, ..., 0)$

$$\left.\begin{cases} \partial_t U = D\Delta U + A_1 U + C \cdot \nabla U + (A_1 - A_2) Y_2, \text{ in } (0,T) \times \Omega \\ U = 0, \text{ on } (0,T) \times \partial\Omega \\ \\ u_1 = 0, \text{ in } (0,T) \times \omega \end{cases}\right\} \overset{?}{\Rightarrow} A_1 = A_2$$

Lipschitz Satbility

- $$\|A_1 - A_2\|_{L^2(\Omega)} \leq C\Big(\|B(Y_1 - Y_2)\|_{(0,T)\times\omega} + \ldots ?\Big)$$

- Carleman estimates for

$$\begin{cases} \partial_t U = D\Delta U + AU + C\cdot\nabla U + F \\ U_{|\partial\Omega} = 0 \end{cases}$$

Carleman Estimates for parabolic equations

A. Fursikov-O. Yu. Imanuvilov, Seoul National University, 96

$$P = \partial_t - \Delta_x, \text{ on } \Omega_T = (0, T) \times \Omega$$
$$\omega \subset\subset \Omega$$

$$\beta \in W^{2,\infty}(\Omega), \quad \partial_\nu \beta < 0 \text{ on } \partial\Omega, \quad |\nabla \beta| \neq 0 \text{ on } \overline{\Omega \setminus \omega},$$
$$a(t) = \big(t(T-t)\big)^{-1}, \ \varphi(x) = e^{\lambda\beta(x)}, \quad \eta(x) = e^{\lambda\beta(x)} - e^{\lambda\overline{\beta}} < 0.$$
$$Pu = f$$

$$s^3\lambda^4 \|(a\varphi)^{\frac{3}{2}} e^{sa\eta} u\|^2_{L^2(\Omega_T)} + s\lambda^2 \|(a\varphi)^{\frac{1}{2}} e^{sa\eta} \nabla_x u\|^2_{L^2(\Omega_T)} \leq C\Big(\|e^{sa\eta} f\|^2_{L^2(\Omega_T)} + s^3\lambda^4 \|(a\varphi)^{\frac{3}{2}} e^{sa\eta} u\|^2_{L^2(\omega_T)}\Big),$$

for $s \geq s_0, \quad \lambda \geq \lambda_0, \ u \in \mathcal{C}^\infty(\overline{\Omega_T})$, and $u_{|\partial\Omega} = 0$.

Carleman Estimates and Inverse Problems

- A.L. Bukhgeim, M.V. Klibanov, Soviet. Math. Dokl, 81
  Hölder stability results with local Carleman estimates.
- 0. Yu. Imanuvilov, M. Yamamoto, I.P, 98
  Lipschitz stability results with global Carleman estimates.
  Parabolic problems.

Carleman Estimates and Inverse Problems

Imanuvilov-Yamamoto (98) : **Potentials identification.**

$$\begin{cases} y' - \Delta y + a(x) y = 0 & \text{in} \quad (0,T) \times \Omega, \\ y = h, & \text{on} \quad (0,T) \times \Omega, \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases}$$

$$n = 1, \quad A = (a)$$

$$\theta \in (0,T), \quad y_2(\theta, .) \geq \delta > 0$$

$$\exists\, C > 0,\ \|a_2 - a_1\|^2_{L^2(\Omega)} \leq C \left(\|(y_2 - y_1)(\theta, .)\|^2_{H^2(\Omega)} + \|y_2 - y_1\|^2_{H^1(0,T;L^2(\omega))} \right)$$

In particular, if $y_1(\theta, .) = y_2(\theta, .)$:

$$\|a_2 - a_1\|^2_{L^2(\Omega)} \leq C \|y_2 - y_1\|^2_{H^1(0,T;L^2(\omega))}$$

- Starting with the pioneer work of Bukhgeim-Klibanov, Carleman estimates have been successfully used for the uniqueness and stability for determining coefficients.
- Often, it is difficult to observe all the components of parabolic systems.
- In the results presented here, we only observe one component and we need the knowledge of the solution at a fixed time $\theta \in (0, T)$

Identification of coefficients of a 2 × 2 parabolic systems

M. Cristofol, P. Gaitan and H. Ramoul, I.P, 06

- $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$

$$\begin{cases} \partial_t y_1 = d_1 \Delta y_1 + a(x) y_1 + b(x) y_2 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t y_2 = \Delta y_2 + c(x) y_1 + d(x) y_2 & \text{in} \quad 0,T) \times \Omega,, \\ y_1 = h_1, \; y_2 = h_2 & \text{on} \quad (0,T) \times \partial\Omega, \\ y_1(0) = y_{10} \;\; \text{et} \;\; y_2(0) = y_{20} & \text{in} \quad \Omega, \end{cases}$$

- $a, b, c, d \in \Lambda(R) = \{\Phi \in L^\infty(\Omega); \; \|\Phi\|_{L^\infty(\Omega)} \leqslant R\}$
- $$\widetilde{y}_{10} \geq r, \; \widetilde{y}_{20} \geq r,$$
- $$b_{|\omega} \geq b_0 > 0$$

Applications : medecine, biology, ecology ...

Stability Theorem

Theorem (M. Cristofol, P. Gaitan, H. Ramoul, I.P, 06)

Under the previous assumptions, there exists a constant $C > 0$ such that

$$\begin{aligned} \|b-\widetilde{b}\|^2_{L^2(\Omega)} + \|c-\widetilde{c}\|^2_{L^2(\Omega)} &\leq C\Big(\|\partial_t y_1 - \partial_t \widetilde{y}_1\|^2_{L^2(\omega_T)} \\ &+\|(y_1, y_2)(\theta) - (\widetilde{y}_1, \widetilde{y}_2)(\theta)\|_{H^2(\Omega)}\Big) \end{aligned}$$

$$\begin{cases} \partial_t y_1 = \alpha\Delta y_1 + a(x)y_1 + b(x)y_2 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t y_2 = \Delta y_2 + c(x)y_1 + d(x)y_2 & \text{in} \quad 0,T)\times\Omega,, \\ y_1 = h_1,\ y_2 = h_2 & \text{on} \quad (0,T)\times\partial\Omega, \\ y_1(0) = y_{10} \ \text{ et } \ y_2(0) = y_{20} & \text{in} \quad \Omega, \end{cases}$$

Linearized Inverse Problem

Let (y_1, y_2) and $(\widetilde{y_1}, \widetilde{y_2})$ be solutions to

$$\begin{cases} \partial_t y_1 = d_1 \Delta y_1 + a y_1 + b y_2 & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + c y_1 + d y_2 & \text{in } Q_T, \\ y_1 = h_1, \; y_2 = h_2 & \text{on } \Sigma_T, \\ y_1(0) = y_{10} \quad \text{et} \quad y_2(0) = y_{20} & \text{in } \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \widetilde{y_1} = d_1 \Delta \widetilde{y_1} + a \widetilde{y_1} + \widetilde{b} \widetilde{y_2} & \text{in } Q_T, \\ \partial_t \widetilde{y_2} = \Delta \widetilde{y_2} + \widetilde{c} \widetilde{y_1} + d \widetilde{y_2} & \text{in } Q_T, \\ \widetilde{y_1} = h_1, \; \widetilde{y_2} = h_2 & \text{on } \Sigma_T, \\ \widetilde{y_1}(0) = \widetilde{y}_{10} \quad \text{et} \quad \widetilde{y_2}(0) = \widetilde{y}_{20} & \text{in } \Omega. \end{cases}$$

Denote $U_1 = y_1 - \widetilde{y_1}$, $U_2 = y_2 - \widetilde{y_2}$, $V_1 = \partial_t (y_1 - \widetilde{y_1})$, $V_2 = \partial_t (y_2 - \widetilde{y_2})$, $\gamma_1 = b - \widetilde{b}$ and $\gamma_2 = c - \widetilde{c}$.
Then (V_1, V_2) is solution to

$$\begin{cases} \partial_t V_1 = d_1 \Delta V_1 + a U_1 + b V_2 + \gamma_1 \partial_t \widetilde{y_2} & \text{in } Q_T, \\ \partial_t V_2 = \Delta V_2 + c V_1 + d V_2 + \gamma_2 \partial_t \widetilde{y_1} & \text{in } Q_T, \\ V_1 = V_2 = 0 & \text{on } \Sigma_T \end{cases}$$

Main argument of the proof

Local Carleman estimate

$$b_{|\omega} \geq b_0 > 0,$$
$$\omega \subset \omega' \quad s' < s$$

- F. Ammar Khodja, AB and C. Dupaix, J.M.A.A, 06

$$\|(a\varphi)^{\frac{3}{2}} e^{sa\eta} V_2\|_{L^2(\omega_T)} \leq C\Big(\|(a\varphi)^{\frac{3}{2}} e^{s' a\eta} V_1\|_{L^2(\omega'_T)} + \|e^{s' a\eta}\gamma_1 \partial_t \widetilde{y}_2\|_{L^2(\Omega_T)} + \|e^{s' a\eta}\gamma_2 \partial_t \widetilde{y}_1\|_{L^2(\Omega_T)}\Big)$$

- A more accurate observability estimate
  M. Cristofol, P. Gaitan and H. Ramoul, I.P, 06

$$\|(a\varphi)^{\frac{3}{2}} e^{sa\eta} V_2\|_{L^2(\omega_T)} \leq C\Big(s^4\lambda^4\|(a\varphi)^{\frac{7}{2}} e^{sa\eta} V_1\|_{L^2(\omega'_T)} + \|e^{sa\eta}\gamma_1 \partial_t \widetilde{y}_2\|_{L^2(\Omega_T)} + \|e^{sa\eta}\gamma_2 \partial_t \widetilde{y}_1\|_{L^2(\Omega_T)}\Big)$$

2×2 Reaction-Diffusion-Convection Systems

Joint work with

1. M. Cristofol, P. Gaitan, M. Yamamoto, Applicable Analysis, 2009
2. M. Cristofol, P. Gaitan, L. De Teresa, submitted to SICON, 2010

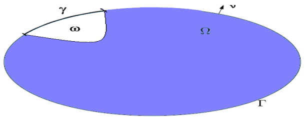
2 × 2 Reaction-Diffusion-Convection Systems

$$\begin{cases} \partial_t y_1 = d_1 \Delta y_1 + a y_1 + b y_2 + A \cdot \nabla y_1 + B \cdot \nabla y_2 \quad \text{in} \quad (0,T) \times \Omega \\ \partial_t y_2 = \Delta y_2 + c y_1 + d y_2 + C \cdot \nabla y_1 + D \cdot \nabla y_2 \quad \text{in} \quad (0,T) \times \Omega \\ y_1 = h_1, \; y_2 = h_2 \quad \text{on} \quad (0,T) \times \partial\Omega, \\ y_1(0) = y_{10}, \; y_2(0) = y_{20} \quad \text{in} \quad \Omega, \end{cases}$$

Inverse Problem

Determine $b(x), c(x)$ **from** $y_1|_{\omega_T}$ **and** $(y_1, y_2)|_{\{\theta\}\times\Omega}$

Assumptions



- All the coefficients are in $C^2(\overline{\Omega})$
- $\|b\|_{L^\infty}, \|\widetilde{b}\|_{L^\infty}, \|c\|_{L^\infty}, \|\widetilde{c}\|_{L^\infty} \leq M$ fixed constant
- $\|\widetilde{y}_1\|_{C(\overline{\Omega_T})}, \ \|\widetilde{y}_2\|_{C(\overline{\Omega_T})}, \|\widetilde{y}_1\|_{C^3(\omega_T)}, \ \|\widetilde{y}_2\|_{C^3(\omega_T)} \leq M$
- $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$
- $|B(x) \cdot \nu(x)| \neq 0$ on γ, ν: unit outward normal to $\partial\Omega$
- $|\widetilde{y}_1(\cdot, \theta)| \geq \delta_0 > 0, |\widetilde{y}_2(\cdot, \theta)| \geq \delta_0 > 0, \quad \theta \in (0, T).$

Stability Result

Theorem (Benabdallah, Cristofol, G., De Teresa, 10)

*Under the previous assumptions and if b **is known in** ω, then there exists a constant $\kappa > 0$ such that*

$$
\begin{aligned}
\|b-\widetilde{b}\|_{L^2(\Omega)} + \|c-\widetilde{c}\|_{L^2(\Omega)} \leq \kappa\Big(&\|y_1-\widetilde{y}_1\|_{H^1(0,T;L^2(\omega))} \\
&+\|(y_1,y_2)(\theta)-(\widetilde{y}_1,\widetilde{y}_2))(\theta)\|_{H^2(\Omega)}\Big)
\end{aligned}
$$

2×2 Reaction-Diffusion-Convection Systems

- $(\widetilde{y}_1, \widetilde{y}_2)$ solution to the previous system where b, c are replaced by $\widetilde{b}, \widetilde{c}$.
- $u = y_1 - \widetilde{y}_1$, $v = y_2 - \widetilde{y}_2$
- $f = (b - \widetilde{b})\widetilde{y}_1, \quad g = (c - \widetilde{c})\widetilde{y}_2$

(u, v) solution to

$$\begin{cases} \partial_t u = d_1 \Delta u + au + bv + A \cdot \nabla u + B \cdot \nabla v + f, & \text{in } (0,T) \times \Omega \\ \partial_t v = \Delta v + cu + dv + C \cdot \nabla u + D \cdot \nabla v + g, & \text{in } (0,T) \times \Omega \\ u = v = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0) = u_0, \ v(0) = v_0 & \text{in } \Omega \end{cases}$$

Carleman estimates for (u, v) with the observation of $u_{|(0,T)\times\omega}$ and $(f, g)_{|(0,T)\times\Omega}$?

Local Carleman estimate

Lemma

...then, there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ *and* $C = C(\Omega, \omega, T)$ *such that for all* $\lambda \geq \lambda_1$ *and* $s \geq s_1$

$$\|(a\varphi)^{\frac{3}{2}} e^{sa\eta} v\|_{L^2(\omega_T)} \leq C(s, \lambda, T))\left(\|u\|^2_{L^2(\omega'_T)} + \|f\|^2_{L^2(\omega'_T)} + \|e^{sa\eta}(f, g)\|^2_{L^2(\Omega_T)}\right)$$

$$\omega \subset \omega'$$

$$\begin{cases} \partial_t u = d_1 \Delta u + au + bv + A \cdot \nabla u + B \cdot \nabla v + (b - \widetilde{b})\widetilde{y}_1, & \text{in} \quad (0, T) \times \Omega \\ \partial_t v = \Delta v + cu + dv + C \cdot \nabla u + D \cdot \nabla v + g, & \text{in} \quad (0, T) \times \Omega \\ u = v = 0 & \text{on} \quad (0, T) \times \partial\Omega, \\ u(0) = u_0, \ v(0) = v_0 & \text{in} \quad \Omega \end{cases}$$

Stability Result

Theorem (Benabdallah, Cristofol, G., De Teresa, 10)

*Under the previous assumptions and if b **is known in** ω, then there exists a constant $\kappa > 0$ such that*

$$\|b-\widetilde{b}\|_{L^2(\Omega)}+\|c-\widetilde{c}\|_{L^2(\Omega)} \leq \kappa\Big(\|y_1-\widetilde{y_1}\|_{H^1(0,T;L^2(\omega))} + \|(y_1,y_2)(\theta)-(\widetilde{y_1},\widetilde{y_2}))(\theta)\|_{H^2(\Omega)}\Big)$$

Sketch of the proof

Consider the simplest case where

- $$\Omega = (0,1) \times \Omega', \ \omega = (0,\epsilon) \times \omega', \ \gamma = \{0\} \times \omega'$$
- $$B(x) = (1,0,...,0), \quad x = (x_1, x'),$$
  with
  $$\Omega' \subset \mathbb{R}^{d-1} \qquad \omega' \subset \Omega'$$
- The second equation of the system becomes
  $$\partial_{x_1} v + bv = \partial_t u - \Delta u - au - A \cdot \nabla u - f.$$

Step 1 : The inversion

- **Step 1: An equation for v.**

$$L := \partial_{x_1} + b, \quad D(L) = \{v \in H^1(\omega); v(0, x') = 0, \text{ on } \omega'\}$$

$$L^{-1}(w)(t,x) = e^{\int_0^{x_1} b(y_1,x')dy_1} \int_0^{x_1} e^{-\int_0^{y_1} b(x_1,x')dx_1} w(t,y_1,x')dy_1, \ \forall w \in L^2(\omega$$

$$\begin{aligned} v &= L^{-1}\big(\partial_t u - \Delta u - au - A\cdot\nabla u - f\big) \quad \text{a.e. in } \omega_T \\ &= \mathcal{P}u + \mathcal{Q}f + \partial_{x_1} u(t,0,x'), \end{aligned}$$

where $\mathcal{P}$ is parabolic operator and $\mathcal{Q}$ a bounded operator on $L^2(\omega)$.

Steps 2 and 3

1. Step 2: An observability inequality for v with two observations u on ω_T and $\partial_\nu u$ on $(0,T)\times\gamma$.

$$\int_{\omega_T}(s\varphi)^{\tau_2+3}e^{2sa\eta}|v|^2dx\,dt \quad \leq C(s,\lambda)\Big(\int_{\widetilde{\omega}_T} M(x',t)|u|^2\,dt\,dx' + \|f\|^2_{L^2(\widetilde{\omega}_T)}\Big)$$

$$+\int_{\omega'_T}(s\varphi)^{\tau_2+3}e^{2sa\eta}|\partial_{x_1}u(0,x',t)|^2\,dt\,dx',$$

with $M(x',t)=\int_0^1\varphi^{\tau_2+7}e^{2sa\eta}dx_1$.
It remains to estimate the boundary term.

2. Step 3: Estimates of the boundary term. By choosing a suitable weight β and... some technical computations....

3 × 3 parabolic system

1. M. Cristofol, P. Gaitan, M. Yamamoto, Applicable Analysis, 2009
2. M. Cristofol, P. Gaitan, L. De Teresa, submitted to SICON, 2010

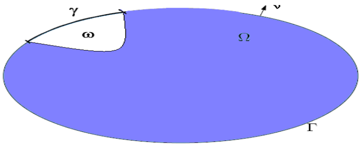
$$\begin{cases} \partial_t y_1 = d_1 \Delta y_1 + a_{11} y_1 + a_{12} y_2 + a_{13} y_3 + f_1 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t y_2 = d_2 \Delta y_2 + a_{21} y_1 + a_{22} y_2 + a_{23} y_3 + f_2 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t y_3 = d_3 \Delta y_3 + a_{31} y_1 + a_{32} y_2 + a_{33} y_3 + f_3 & \text{in} \quad \Omega, \\ y_1 = h_1,\ y_2 = h_2,\ y_3 = h_3 & \text{on} \quad (0,T)\times\partial\Omega, \\ y_1(\cdot,0) = y_{10},\ y_2(\cdot,0) = y_{20},\ y_3(\cdot,0) = y_{30}, & \text{in } \Omega. \end{cases}$$

$$(a_{ij})_{1\le i\le j\le 3} \subset L^\infty(\Omega)$$

Is it possible to recover some coefficients of A with the knowledge of

$$y_1|_{(0,T)\times\omega} ??$$

Assumptions



1. $(a_{ij})_{1 \leq i,j \leq 3} \in C^4(\overline{\Omega})$
2. $\omega \subset \Omega$ neighborhood of Ω
3. $d_2 = d_3$
4. There exists $j \in \{2, 3\}$ such that $|a_{1j}(x)| \geq C > 0$ for all $x \in \omega$ and for $k_j = \dfrac{6}{j}$

$$|\left(\nabla a_{1k_j} - \frac{a_{1k_j}}{a_{1j}} \nabla a_{1j}\right) \cdot \nu(x))| \neq 0, \text{ on } \gamma = \partial\omega \cap \partial\Omega$$

A simple example

$\Omega = (0,1)$, $\omega = (0,a)$ with $0 < a < 1$ and $\gamma = \{0\}$,

$$A = \begin{pmatrix} a_{11} & x & x+1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The previous assumptions are satisfied

1. $j = 3$ and $a_{13} = x + 1 \neq 0$
2. $k_3 = 2$ and

$$(a_{12}'(0) - \frac{a_{12}}{a_{13}}(0)(a_{13}'(0)) = -2 \neq 0$$

The Lipschitz stability result

$$\left\{\begin{array}{lll} \partial_t y_1 = d_1 \Delta y_1 + a_{11} y_1 + a_{12} y_2 + a_{13} y_3 & \text{in} & (0,T)\times\Omega, \\ \partial_t y_2 = \Delta y_2 + a_{21} y_1 + a_{22} y_2 + a_{23} y_3 & \text{in} & (0,T)\times\Omega, \\ \partial_t y_3 = \Delta y_3 + a_{31} y_1 + a_{32} y_2 + a_{33} y_3 & \text{in} & (0,T)\times\Omega, \\ y_1 = h_1,\ y_2 = h_2,\ y_3 = h_3 & \text{on} & (0,T)\times\partial\Omega, \\ y_1(.,0) = y_{10} \quad y_2(.,0) = y_{20} \ \text{ and } \ y(.,0) = y_{30} & \text{in} & \Omega. \end{array}\right. \tag{5.4}$$

Theorem (A.B, M. Cristofol, P. Gaitan, L. de Teresa, submitted 09)

Assume

1. *Previous assumptions*
2. *Smoothness assumptions*
3. $(\widetilde{a}_{ij})$ *are such that there exist* $C > 0$ *and* $\theta \in (0, T)$ *such that* $|\widetilde{y}_1(\theta, .)| \geq C, \quad |\widetilde{y}_2(\theta, .)| \geq C, \quad |\widetilde{y}_3(\theta, .))| \geq C.$
4. $a_{ij} = \widetilde{a}_{ij}$ *on* ω *for* $(i, j) \in \{(2, 1), (3, 2), (1, 3)\}$

Then there exists $\kappa > 0$ *such that*

$$\|a_{12} - \tilde{a}_{12}\|^2_{L^2(\Omega)} + \|a_{23} - \tilde{a}_{23}\|^2_{L^2(\Omega)} + \|a_{31} - \tilde{a}_{31}\|^2_{L^2(\Omega)} \leq \kappa \Big(\|\partial_t(y_1 - \widetilde{y}_1)\|^2_{L^2(\omega_T)} + \|(y_1 - \widetilde{y}_1)(\theta)\|^2_{H^2(\Omega)} + \|(y_2 - \widetilde{y}_2)(\theta)\|^2_{H^2(\Omega)} + \|(y_3 - \widetilde{y}_3)(\theta)\|^2_{H^2(\Omega)} \Big) .$$

The main argument : A Carleman type Estimate

Theorem (A.B, M. Cristofol, P. Gaitan, L. de Teresa, submitted 09)

Under the previous assumptions, there exists a positive function $\beta \in C^2(\overline{\Omega})$ such that the following Carleman estimate holds for all solutions of the previous system

$$\|(a\varphi)^{\frac{3}{2}} e^{sa\eta} Y\|^2_{L^2(\Omega_T)} + \|(a\varphi)^{\frac{1}{2}} e^{sa\eta} \nabla_x Y\|^2_{L^2(\Omega_T)}$$

$$\leq C_1(s,\lambda,T)\left(\|y_1\|^2_{L^2(\omega_T)} + \|(f_2,f_3)^t\|^2_{L^2(\omega_T)}\right) + \|e^{sa\eta}(f_1,f_2,f_3)^t\|^2_{L^2(\Omega_T)}$$

where

$$Y = (y_1, y_2, y_3)^t$$

Reduction to a 2 × 2 parabolic systems with convection terms

- $$z = a_{12}y_2 + a_{13}y_3 \quad \text{in } \omega_T. \tag{5.5}$$
- Suppose, for example, that the previous assumptions are satisfied for $j = 3$
- $$\begin{cases} \partial_t y_1 = d_1 \Delta u_1 + a_{11} y_1 + z + f & \text{in} \quad \omega_T, \\ \partial_t z = \Delta z + A \cdot \nabla z + az + ey_1 + B \cdot \nabla y_3 + by_3 + G & \text{in} \quad \omega_T, \\ \partial_t y_3 = \Delta y_3 + a_{31} y_1 + a_{32} y_2 + a_{33} y_3 & \text{in} \quad (0, T) \times \Omega \end{cases}$$
- $Ly_3 := B \cdot \nabla y_3 + by_3 = \partial_t z - \Delta z - A \cdot \nabla z - az - ey_1 - G$
- Use the previous Carleman estimate

1. $(a_{ij})_{1\leq i,j\leq 3} \in C^4(\overline{\Omega})$
2. $\omega \subset \Omega$ neighborhood of Ω
3. $d_2 = d_3$
4. There exists $j \in \{2,3\}$ such that $|a_{1j}(x)| \geq C > 0$ for all $x \in \omega$ and for $k_j = \dfrac{6}{j}$

$$|\left(\nabla a_{1k_j} - \frac{a_{1k_j}}{a_{1j}}\nabla a_{1j}\right)\cdot \nu(x))| \neq 0, \text{ on } \gamma = \partial\omega \cap \partial\Omega$$

$$B = \left(\nabla a_{12} - \frac{a_{12}}{a_{13}}\nabla a_{13}\right)$$

The finite dimensional case: Kalman observability condition

$A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

$$\left\{ \begin{array}{l} Y' = AY \\ \\ BY = 0, \end{array} \right\} \Rightarrow Y = 0$$

if and only if : $\mathrm{rank}\ [A^* \mid B^*] = \mathrm{rank}\ \left[B^*, A^*B^*, ..., (A^*)^{n-1}B^*\right] = n$

Kalman operator : $\mathcal{K} := \left[B^*, A^*B^*, ..., (A^*)^{n-1}B^*\right] \in \mathcal{L}(\mathbb{R}^{nm}, \mathbb{R}^n)$

$\mathrm{rank}\ [A^* \mid B^*] = n \Leftrightarrow \mathrm{Ker}(\mathcal{K}^*) = \{0\} \Leftrightarrow \ker B \cap \{\text{eigenvectors of } A\} = \emptyset$

The infinite dimensional case for constant coefficients

$$\mathcal{B} = B1_\omega$$

where :

$$B \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \quad \omega \subset\subset \Omega,$$

$$L := I_d \Delta + A, \quad Y' = LY$$

A, B matrices with constant coefficient

Theorem (Kalman condition, F. Ammar Khodja, A.B, C. Dupaix, M. González-Burgos, J.E.E, 09)

$(L, \mathcal{B})$ *is observable if and only if* $(\mathcal{L}^*, \mathcal{B}^*)$ *is controllable if and only if*

$$\text{rank } [A^* \mid B^*] = n$$

1. Through a control argument, we can construct solutions $\widetilde{Y}$ satisfying the positivity assumption.
2. We can determine all the coefficients of the system. For it, we need to repeat the measurements.
3. The results generalize to $n \times n$ reaction-diffusion systems, with $n-2$ observations : Joint work with M. Cristofol, P. Gaitan and L. de Teresa
4. Boundary Observability : A new result by F. Ammar Khodja, A.B, M. González-Burgos, L. de Teresa, almost submitted, 2011 for the one dimensional case and constants coefficients matrices. But no Carleman estimate.
   Inverse problem is an open question.

Thank you for attention