Stable Determination of the Electromagnetic Coefficients by Boundary Measurements

Pedro Caro

Outline

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DBV problem

- Electromagnetic fields will be assumed to be time-harmonic

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Ω bounded Lipschitz domain and $\mu, \varepsilon, \sigma \in L^\infty(\Omega)$ with

$$\mu(x) \geq \mu' > 0, \quad \varepsilon(x) \geq \varepsilon' > 0, \quad \sigma(x) \geq 0.$$

a. e. in Ω. Given $T \in TH(\partial\Omega)$, the problem

$$\text{find } E, H \in H(\Omega; \operatorname{curl}) \text{ solving ME in } \Omega \text{ with } N \times E = T$$

is well-posed for $\omega \in \mathbb{C} \setminus F$. F has no accumulation point in $\mathbb{C} \setminus \{0\}$.

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- $\omega \in F$ is called resonant frequency.

IBV problem

Boundary measurements can be modeled by **Admittance map**:

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where $N \times E = T$ with E, H the solution for

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writing $\gamma = \varepsilon + i\sigma/\omega$ to be more concise.

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Inverse problem: recover μ, γ from Λ.

- *Uniqueness*: $\mu_j, \gamma_j \in L^\infty(\Omega)$ and Λ_j their corresponding admittance map $j = 1, 2$.

$$\Lambda_1 = \Lambda_2 \Longrightarrow \mu_1 = \mu_2,\ \gamma_1 = \gamma_2?$$

- *Stability*: Is there a modulus of continuity b such that

$$\|\mu_1 - \mu_2\|_{L^\infty(\Omega)} + \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq b(\|\Lambda_1 - \Lambda_2\|)?$$

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Same kind of problem can be proposed from partial knowledge of Λ.

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- **Pseudo-metric distance**:

$$\delta(C_1, C_2) = \max_{j \neq k} \sup_{\substack{(T_k, S_k) \in C_k \\ \|T_k\|_{TH(\partial\Omega)} = 1}} \inf_{(T_j, S_j) \in C_j} \|(T_j, S_j) - (T_k, S_k)\|_{(TH(\partial\Omega))^2}.$$

  - $\delta(C_1, C_2) = 0 \Longrightarrow \overline{C_1} = \overline{C_2}$.
  - When ω is a non-resonant for μ_j, γ_j

$$\delta(C_1, C_2) \leq \|\Lambda_1 - \Lambda_2\| \leq C\delta(C_1, C_2).$$

Outline

Admissible class of coefficients

Admissible: Given $0 < M$, $0 < s < 1/2$, μ, γ is admissible if

(i) *ellipticity*, $\gamma, \mu \in C^{1,1}(\overline{\Omega})$ with $M^{-1} \leq \operatorname{Re}\gamma(x)$, $M^{-1} \leq \mu(x)$;

(ii) *a priori bound on the boundary*,

$$\|\gamma\|_{C^{0,1}(\partial\Omega)} + \|\mu\|_{C^{0,1}(\partial\Omega)} < M;$$

(iii) *a priori bound in the interior*,

$$\|\gamma\|_{W^{2,\infty}(\Omega)} + \|\mu\|_{W^{2,\infty}(\Omega)} \leq M, \quad \|\gamma\|_{H^{2+s}(\Omega)} + \|\mu\|_{H^{2+s}(\Omega)} \leq M.$$

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B-stable on the boundary: μ, γ is in the class of B-stable on the boundary if

- μ, γ is admissible,
- $\exists$ a modulus of continuity B : $\forall \tilde{\mu}, \tilde{\gamma}$ admissible, one has

$$\|\partial^\alpha(\gamma - \tilde{\gamma})\|_{L^\infty(\partial\Omega)} + \|\partial^\alpha(\mu - \tilde{\mu})\|_{L^\infty(\partial\Omega)} \leq B\left(\delta(C, \tilde{C})\right),$$

with $0 \leq |\alpha| \leq 1$, $C := C(\mu, \gamma)$ and $\tilde{C} := C(\tilde{\mu}, \tilde{\gamma})$.

Stable determination

Theorem

Ω bounded Lipschitz domain, $\omega > 0$. Then, $\exists C = C(M)$ such that, for any μ_1, γ_1 and μ_2, γ_2 in the class of B-stable on the boundary, one has

$$\|\gamma_1 - \gamma_2\|_{H^1(\Omega)} + \|\mu_1 - \mu_2\|_{H^1(\Omega)} \leq C|\log B(\delta(C_1, C_2))|^{-\lambda},$$

for some $0 < \lambda < 2/3s$. Here $C_j := C(\mu_j, \gamma_j)$ with $j = 1, 2$.

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Corollary

Assume $\partial^\alpha \mu_1|_{\partial\Omega} = \partial^\alpha \mu_2|_{\partial\Omega}$, $\partial^\alpha \gamma_1|_{\partial\Omega} = \partial^\alpha \gamma_2|_{\partial\Omega}$, with $0 \leq |\alpha| \leq 1$. Then, $\exists C = C(M)$ such that

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It should be possible

- to prove that any admissible coefficient is in the class of Hölder-stable on the boundary,
- to check –following Mandache's arguments– that our modulus of continuity is optimal.

Outline

Inside out I

$E_1, H_1, F_2, G_2 \in H(\Omega; \mathrm{curl})$ solutions for

$$\begin{cases} \nabla \times H_1 + i\omega\gamma_1 E_1 = 0 \\ \nabla \times E_1 - i\omega\mu_1 H_1 = 0, \end{cases} \qquad \begin{cases} \nabla \times G_2 + i\omega\overline{\gamma_2} F_2 = 0 \\ \nabla \times F_2 - i\omega\mu_2 G_2 = 0, \end{cases}$$

in Ω. Then

$$\left| \int_\Omega i\omega[(\gamma_1 - \gamma_2) E_1 \cdot \overline{F_2} - (\mu_1 - \mu_2) H_1 \cdot \overline{G_2}]\, dV \right| \leq \delta(C_1, C_2)$$
$$\times \|N \times E_1\|_{TH(\partial\Omega)} \left(\|N \times F_2\|_{TH(\partial\Omega)} + \|N \times G_2\|_{TH(\partial\Omega)} \right).$$

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The procedure consists in:

- constructing exponential growing solutions (EGS for short),

$$E = e^{i\zeta \cdot x}(E_1(\zeta) + E_0(\zeta) + E_{-1}(\zeta))$$
$$H = e^{i\zeta \cdot x}(H_1(\zeta) + H_0(\zeta) + H_{-1}(\zeta))$$

- plugging these solutions in the above estimate,
- getting the estimate for the stability.

Inside out II

After some computations and an interpolation argument we end up with

$$\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \leq C \left(B\big(\delta_C(C_1, C_2)\big) e^{c\tau} + \tau^{2/3 s_1} \right)^{\theta}.$$

where $\tau > 1$, $s_1 < 0$ and

$$f = \gamma_1^{-1/2} \left[\Delta(\gamma_1^{1/2} - \gamma_2^{1/2}) + q_f(\gamma_1^{1/2} - \gamma_2^{1/2}) + p_f(\mu_1^{1/2} - \mu_2^{1/2}) \right],$$
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Using a Carleman estimate with boundary terms we get

$$\|\gamma_1 - \gamma_2\|_{H^1(\Omega)} + \|\mu_1 - \mu_2\|_{H^1(\Omega)} \leq$$
$$\leq C e^{\frac{d_2 - d_1}{2h}} \left(B\big(\delta_C(C_1, C_2)\big) e^{c\tau} + \tau^{2/3 s_1} \right)^{\frac{s_2}{s_2 - s_1}} + C e^{\frac{d_2 - d_1}{2h}} B\big(\delta_C(C_1, C_2)\big),$$

where $d_2 > d_1$, $s_1 < 0 < s_2 < 1/2$, τ large enough and h small enough.

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where $d_2 > d_1$, $s_1 < 0 < s_2 < 1/2$, τ large enough and h small enough. In order to obtain the stability, choose

$$\tau = -\frac{1}{2c} \log B\big(\delta_C(C_1, C_2)\big).$$

Thank you for your attention!