Inverse Problems with Partial Data in a Slab

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(Joint work with G. Uhlmann)

Inverse Problems: Theory and Applications

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Schrödinger Equation in a Slab

$$(-\Delta + q(x) - k^2)u(x) = 0 \quad in\ \Omega$$
$$u(x) = f(x) \quad on\ \partial\Omega$$

- domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is an infinite slab between two parallel hyperplanes Γ_1 and Γ_2.
- potential $q(x) \in L^\infty(\Omega)$ with compact support in $\mathbb{R}^n$.
- $f|_{\Gamma_j}$ has compact support in Γ_j, $j = 1, 2$.
- u satisfies the partial radiation condition introduced by Sveshnikov.

Partial Radiation Condition

WLOG, we assume

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n : x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, 0 < x_n < L\}$$

and

$$\Gamma_1 = \{x \in \mathbb{R}^n : x_n = L > 0\},\ \Gamma_2 = \{x \in \mathbb{R}^n : x_n = 0\}$$

Partial radiation condition reads

$$\left(\frac{\partial}{\partial \rho} - ik_m\right) u_m(x') = o(\rho^{\frac{2-n}{2}}), \quad as\ \rho \to \infty$$

where $u_m(x') = \frac{2}{L}\int_0^L u(x) \sin \frac{m\pi x_n}{L} dx_n$, $k_m = k(1 - \frac{m^2\pi^2}{k^2L^2})^{\frac{1}{2}}$,
$\rho = |x'|$, $m = 1, 2, \cdots$.

The Forward Problem and The Inverse Problem

Theorem: For all k except a discrete set, there exists a unique solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$ such that $f|_{\Gamma_j}$ is compactly supported in Γ_j, $j = 1, 2$.

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The Dirichlet-to-Neumann map:

$$\Lambda_q : f \longrightarrow \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

where ν is the unit outer normal vector.

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Inverse Problem: Determine q from Λ_q.

Inverse Problem with Partial Data:

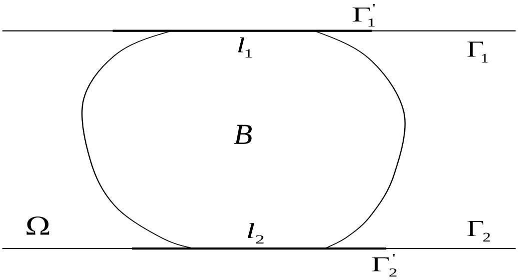
Determine q from only partial knowledge of Λ_q.

Partial Data on the Boundary

Let Γ_1' be any open set on Γ_1 containing the support of $q|_{\Gamma_1}$, and Γ_2' be any open set on Γ_2 containing the support of $q|_{\Gamma_2}$. Define the following two sets of partial boundary measurements:

$$C^D_{q,\,\Gamma_2'} := \{\Lambda_q(f)\big|_{\Gamma_2'} \text{ for all } f \text{ with } \mathrm{supp}(f) \subset \Gamma_1\}$$

$$C^S_{q,\,\Gamma_1'} := \{\Lambda_q(f)\big|_{\Gamma_1'} \text{ for all } f \text{ with } \mathrm{supp}(f) \subset \Gamma_1\}$$



Main Results

Theorem 1

If $C^D_{q_1,\,\Gamma_2'} = C^D_{q_2,\,\Gamma_2'}$, then $q_1(x) = q_2(x)$ in Ω.

Theorem 2

If $C^S_{q_1,\,\Gamma_1'} = C^S_{q_2,\,\Gamma_1'}$, then $q_1(x) = q_2(x)$ in Ω.

Results on Partial Data Inverse Boundary Value Problems

- Bukhgeim and Uhlmann (2002): Half of the boundary
- Kenig, Sjöstrand and Uhlmann (2007): Small set of the boundary
- Isakov (2007): Special geometry
- Nachman and Street (2010): Reconstruction
- Imanuvilov, Uhlmann and Yamamoto (2010): Two dimensions
- Others...

Proof of Theorem 1

Key identity:

$$\int_{\Omega}(q_1 - q_2)u_1 u_2 dx = -\int_{l_1} \frac{\partial w}{\partial \nu} u_2 ds$$

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where

$$\begin{cases} (-\Delta + q_1(x) - k^2)u_1(x) = 0 \quad in\,\Omega \\ u_1(x) = 0 \quad on\,\Gamma_2 \end{cases}$$

$$(-\Delta + q_2(x) - k^2)u_2(x) = 0 \quad in\,\Omega$$

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$$w(x) = v(x) - u_1(x)$$

$$\begin{cases} (-\Delta + q_2(x) - k^2) v(x) = 0 \quad in\ \Omega \\ v(x) = u_1(x) \quad on\ \Gamma_1 \cup \Gamma_2 \end{cases}$$

Carleman Estimate

Carleman estimate:

$$\tau \int_{I_1} (\eta \cdot \nu) \, |e^{-\tau x \cdot \eta} \frac{\partial w}{\partial \nu}|^2 ds \;\; \leq \;\; \int_{\Omega \cap B} |e^{-\tau x \cdot \eta} (-\Delta + q_2 - k^2) w|^2 dx$$

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where τ is a large parameter.

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$$= \int_{\Omega\cap B} |e^{-\tau x\cdot\eta}(q_1 - q_2)u_1|^2 dx$$

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Choose $\eta = (\eta_1, \eta_2, \eta_3)$ such that $\eta \cdot \nu = \eta_3 > 0$ on $l_1 \subset \Gamma_1$.

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Key inequality:

$$
\begin{aligned}
&\left|\int_{\Omega} (q_1 - q_2) u_1 u_2 dx\right| \\
\le\ & \left(\frac{1}{\tau(\eta\cdot\nu)}\right)^{\frac{1}{2}} \left(\int_{\Omega\cap B} |e^{-\tau x\cdot\eta}(q_1 - q_2)u_1|^2 dx\right)^{\frac{1}{2}} \left(\int_{l_1} |e^{\tau x\cdot\eta} u_2|^2 ds\right)^{\frac{1}{2}}
\end{aligned}
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Construction of Complex Geometrical Optics Solutions

For u_2:

$$(-\Delta + q_2(x) - k^2)u_2(x) = 0 \quad in\ \Omega$$

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$$u_2(x) = e^{x\cdot\rho_2}(1 + \psi_2(x,\rho_2)), \quad (\rho_2 \cdot \rho_2 = 0)$$

$\psi_2(x,\rho_2)$ goes to 0 as $|\rho_2| \to \infty$.

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$\psi_2(x, \rho_2)$ goes to 0 as $|\rho_2| \to \infty$.

$$u_1(x) = e^{x\cdot\rho_1}(1 + \psi_1(x, \rho_1)) - e^{x^*\cdot\rho_1}(1 + \psi_1(x^*, \rho_1)), \quad (\rho_1 \cdot \rho_1 = 0)$$

Do even extension about x_3 for $q_1(x)$ and denote $x^* = (x_1, x_2, -x_3)$.
$\psi_1(x, \rho_1)$ and $\psi_1(x^*, \rho_1)$ go to 0 as $|\rho_1| \to \infty$.

Phase Functions

For any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ with $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2} > 0$, we introduce

$$e(1) = \frac{1}{\xi_{1e}}(\xi_1, \xi_2, 0), \quad e(3) = (0, 0, 1), \quad e(2)$$

such that $e(1)$, $e(2)$ and $e(3)$ form a orthogonal normal basis in $\mathbb{R}^3$.

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Denote the coordinate of $x \in \mathbb{R}^3$ in this basis by $(x_{1e}, x_{2e}, x_{3e})_e$.
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$$\xi = (\xi_{1e}, 0, \xi_3)_e$$

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We have

$$\xi = (\xi_{1e}, 0, \xi_3)_e$$

For $\tau >> 0$, we choose

$$\rho_1 = (\frac{i}{2}\xi_{1e} - \tau\xi_3, i|\xi|\sqrt{\tau^2 - \frac{1}{4}}, \frac{i}{2}\xi_3 + \tau\xi_{1e})_e$$

$$\rho_2 = (\frac{i}{2}\xi_{1e} + \tau\xi_3, -i|\xi|\sqrt{\tau^2 - \frac{1}{4}}, \frac{i}{2}\xi_3 - \tau\xi_{1e})_e$$

Identification of the Potential

For $\tau >> 0$, we choose

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- $\rho_1 \cdot \rho_1 = \rho_2 \cdot \rho_2 = 0$
- $\rho_1 + \rho_2 = i\xi$
- $\mathrm{Re}\,(x^* \cdot \rho_1 + x \cdot \rho_2) = -2\tau x_3 \xi_{1e}$
- $\eta = (-\xi_3, 0, \xi_{1e})_e$

$$\left| \int_\Omega e^{ix\cdot\xi}(q_1 - q_2)dx \right| = 0 \Longrightarrow q_1(x) - q_2(x) = 0 \quad in\,\Omega$$

Proof of Theorem 2

Key identity:

$$\int_{\Omega}(q_1 - q_2)u_1 u_2 dx = -\int_{l_2} \frac{\partial w}{\partial \nu} u_2 ds$$

where

$$\begin{cases} (-\Delta + q_1(x) - k^2)u_1(x) = 0 \quad in\, \Omega \cap B \\ u_1(x) = 0 \quad on\, \Gamma_2 \end{cases}$$

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$$(-\Delta + q_2(x) - k^2) u_2(x) = 0 \quad in\, \Omega \cap B$$

If we also require that $u_2(x) = 0$ on Γ_2, then we have the orthogonality relation

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0$$

Identification of the Potential

Phase functions:

$$\rho_1 = (\frac{i}{2}\xi_{1e} - i\tau\xi_3, -|\xi|\sqrt{\tau^2 + \frac{1}{4}}, \frac{i}{2}\xi_3 + i\tau\xi_{1e})_e$$

$$\rho_2 = (\frac{i}{2}\xi_{1e} + i\tau\xi_3, |\xi|\sqrt{\tau^2 + \frac{1}{4}}, \frac{i}{2}\xi_3 - i\tau\xi_{1e})_e$$

Complex geometrical optics solutions:

$$u_j(x) = e^{x\cdot\rho_j}(1 + \psi_j(x, \rho_j)) - e^{x^*\cdot\rho_j}(1 + \psi_j(x^*, \rho_j)), \quad j = 1, 2$$

Identification:

$$q_1 - q_2 = 0 \text{ in } \Omega$$

Electrical Impedance Tomography in a Slab

The conductivity equation:

$$\begin{cases} \mathrm{div}(\gamma \nabla u) = 0 & \text{in} \quad \Omega \\ u(x) = f(x) & \text{on} \quad \partial\Omega \end{cases}$$

$\gamma \in C^2(\bar{\Omega})$, $\gamma > 0$, and $\gamma = 1$ outside a compact set.
The Dirichlet-to-Neumann map is given by

$$\Lambda_\gamma : f \in H^{1/2}(\partial\Omega) \longrightarrow \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$$

Partial Data:

Let Γ_1' be any open set on Γ_1 containing the support of $(\gamma - 1)|_{\Gamma_1}$, and Γ_2' be any open set on Γ_2 containing the support of $(\gamma - 1)|_{\Gamma_2}$.
Define the following two sets of partial boundary measurements:

$$C^D_{\gamma,\,\Gamma_2'} := \{\Lambda_\gamma(f)\big|_{\Gamma_2'} \text{ for all } f \text{ with } \mathrm{supp}(f) \subset \Gamma_1\}$$

$$C^S_{\gamma,\,\Gamma_1'} := \{\Lambda_\gamma(f)\big|_{\Gamma_1'} \text{ for all } f \text{ with } \mathrm{supp}(f) \subset \Gamma_1\}$$

Electrical Impedance Tomography in a Slab

Theorem 3

If $C^D_{\gamma_1,\,\Gamma_2'} = C^D_{\gamma_2,\,\Gamma_2'}$ and

$$\gamma_1 = \gamma_2 \quad \textit{on } \partial\Omega, \qquad \frac{\partial\gamma_1}{\partial\nu} = \frac{\partial\gamma_2}{\partial\nu} \quad \textit{on } \Gamma_2'$$

then $\gamma_1(x) = \gamma_2(x)$ in Ω.

Theorem 4

If $C^S_{\gamma_1,\,\Gamma_1'} = C^S_{\gamma_2,\,\Gamma_1'}$, then $\gamma_1(x) = \gamma_2(x)$ in Ω.

We do not need any further restriction about the conductivity on the boundary in Theorem 4.

Reduction to the Schrödinger Equation

Method: well-known transformation

$$\omega = \gamma^{1/2} u$$

Then ω satisfies

$$(-\Delta + q(x))\omega(x) = 0$$

with

$$q(x) = \gamma^{-1/2}\Delta\gamma^{1/2}$$

THANK YOU!