On some optimal partition problems

Susanna Terracini
Dipartimento di Matematica e Applicazioni
Università di Milano Bicocca

Connection for Women

MSRI–Berkerly, January 13- 14 2011

1 Competition diffusion systems (stationary cases)

- Lotka Volterra systems:

$$-\Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$

- Energy minimizing configurations (superconductors or Bose–Einstein condensates)

$$\mathcal{E}(\psi_1, \cdots, \psi_h) = \int \sum_i \frac{1}{2}|\nabla \psi_i|^2 + F_i(|\psi_i|^2) + \sum_{j\neq i} \beta_{i,j}|\psi_i|^2|\psi_j|^2 \text{ in } \Omega,$$

- Optimal partition problems for Dirichlet eigenvalues:

$$\min \left\{ \sum_{i=1}^{h} \lambda_1^p(\omega_i) \ : \ (\omega_1, \cdots, \omega_k) \in \mathfrak{B}_h(\Omega)) \right\}$$

where

$$\mathfrak{B}_h = \{(\omega_1, \ldots, \omega_h) : \ \omega_i \text{ open}, \ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\}.$$

2 Limiting profiles

As the interspecific competition rate $\beta = \min_{\beta_{ij}}$ tends to infinity we find a vector $U = (u_1, \cdots, u_h)$ of functions

- having mutually disjoint supports: $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$,
- satisfying

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i \neq 0\ , \quad i = 1, \ldots, h,$$

Questions:

- Compactness and a priori bounds
- Uniqueness vs multiplicity
- Extremality conditions
- Regularity

of the minimizers
of the interfaces

3 Segregated critical configuration

Let Ω be an open bounded subset of $\mathbb{R}^N$, with $N \geq 2$. Let $U = (u_1, \ldots, u_h) \in (H^1(\Omega))^h$ be a vector of

- non negative, nontrivial Lipschitz functions in Ω,
- having mutually disjoint supports: $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$,
- satisfying

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i > 0\ , \quad i = 1, \ldots, h,$$

where $f_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are C^1 functions such that $f_i(x,s) = O(s)$ when $s \to 0$, uniformly in x.

Our main interest is the study of the regularity of the nodal set of the segregated configurations $U = (u_1, \ldots, u_h)$:

$$\Gamma_U = \{x \in \Omega : \ U(x) = 0\}$$

Obviously, without other conditions, there is no reason at all why the nodal set should be regular. We must add some information on the interaction between the components at the interface of their supports.

A weak reflection law

Theorem 1 (Tavares-T, 2010) *Let us define, for every* $x_0 \in \Omega$ *and* $r \in (0, dist(x_0, \partial\Omega))$ *the energy*

$$\tilde{E}(r) = \tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 \,,$$

then, $\tilde{E}(x_0, U, \cdot)$ *is an absolutely continuous function of* r*, and we assume that it satisfies the following differential equation*

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(x, u_i)\langle \nabla u_i, x - x_0 \rangle.$$

Then, there exists a set $\Sigma_U \subseteq \Gamma_U$ *the regular part, relatively open in* Γ_U*, such that*

- $\mathcal{H}_{dim}(\Gamma_U \setminus \Sigma_U) \leq N - 2$*, and if* $N = 2$ *then actually* $\Gamma_U \setminus \Sigma_U$ *is a locally finite set;*
- Σ_U *is a collection of hyper-surfaces of class* $C^{1,\alpha}$ *(for every* $0 < \alpha < 1$*). Furthermore for every* $x_0 \in \Sigma_U$

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^\pm$ *are taken from the opposite sides of the hyper-surface. Furthermore, if* $N = 2$ *then* Σ_U *consists in a locally finite collection of curves meeting with equal angles at singular points.*

5 Some remarks

$$\frac{d}{dr}\frac{1}{r^{N-2}}\int_{B_r(x_0)}|\nabla U|^2=\frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}(\partial_\nu U)^2\,d\sigma+\frac{2}{r^{N-1}}\int_{B_r(x_0)}\sum_i f_i(x,u_i)\langle\nabla u_i,x-x_0\rangle. \quad \text{(WRL)}$$

- It is easily checked that equation (WRL) always holds for balls lying entirely inside one of the component supports, as a consequence of the elliptic equation. Hence, for our class systems, it represents the only interaction between the different components u_i through the common boundary of their supports;
- (WRL) is satisfied by the nodal components of solutions to a single semilinear elliptic equation of the form $-\Delta u = f(u)$.
- equation (WRL) can be seen as a weak form of a reflection property through the interfaces. Consider the following example: take two linear functions on complementary half-spaces:

$$u_1(x)=a_1x_1^+ \qquad u_2(x)=a_2x_1^- \,.$$

Then

$$\text{(WRL)} \qquad \Longleftrightarrow \qquad |a_1|=|a_2|\,.$$

More in general, when we have two components with a smooth interface between the supports, then

$$(\text{WRL}) \qquad \Longleftrightarrow \qquad \lim_{x\to x_0^+} |\nabla U(x)| = \lim_{x\to x_0^-} |\nabla U(x)| \,.$$

(WRL) as an extremality condition

Although this hypothesis may look weird and may seem hard to check in applications, it occurs naturally in many situations where the vector U appears as a limit configuration in problems of spatial segregation.

- It has to be noted indeed that a form of (WRL) always holds for solutions of systems of interacting semilinear equations and that it persists under strong H^1 limits.
- In addition, (WRL) holds for vector functions U minimizing Lagrangian functional associated with the system.
- It is fullfilled also for strong limits to competition–diffusion systems, both those possessing a variational structure and those with Lotka-Volterra type interaction.
- Our theorem extends also to sign changing, complex and vector valued functions u_i. Lipschitz continuity can be weakened into Hölder continuity for every $\alpha \in (0, 1)]$.

6 More remarks

Assume U minimizes a Lagrangian energy with a pointwise constraint of the type $U(x) \in \Sigma$, for almost every $x \in \Omega$. Let $Y \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^N)$. Then, differentiation of the energy with respect to ε with $U(x) \mapsto U_\varepsilon(x) = U(x+\varepsilon Y(x))$ yields the well known identity

$$\int_\Omega \left\{ dY(x)\nabla U(x)\cdot\nabla U(x) - \mathrm{div}Y(x)\left[\frac{1}{2}|\nabla U(x)|^2 - F(U(x))\right]\right\} dx = 0 \;, \forall Y \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^N)\;.$$

By localizing to a regular $\omega \subset \Omega$ this implies

$$\int_\omega \left\{ dY(x)\nabla U(x)\cdot\nabla U(x) - \mathrm{div}Y(x)\left[\frac{1}{2}|\nabla U(x)|^2 - F(U(x))\right]\right\} dx$$
$$= \int_{\partial\omega} \left\{ Y(x)\cdot\nabla U(x)\nu(x)\cdot\nabla U(x) - \nu(x)\cdot Y(x)\left[\frac{1}{2}|\nabla U(x)|^2 - F(U(x))\right]\right\} d\sigma\;,$$
$$\forall \text{ smooth } \omega \text{ and } \forall Y \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^N)\;. \quad (*)$$

Next

$$(*) \quad + \quad \begin{pmatrix} Y(x) & = x - x_0 \\ \omega & = B_r(x_0) \end{pmatrix} \quad \Longrightarrow \quad (WRL)$$

7 Almgren's monotonicity formula

$$E(r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U|^2 - \langle F(x,U), U \rangle \right),$$
$$H(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |U|^2,$$

We define the (modified) Almgren's quotient as follows:

$$N(r) = \frac{E(r) + H(r)}{H(r)}.$$

Theorem 2 *Assume (WRL). Then, there exist* $\bar{r}, C > 0$*, such that for* $0 < r < \bar{r}$ *we have* $H(r) \neq 0$ *and also* $E(r) + H(r) \neq 0$*,*

$$N'(r) \geq -CrN(r)$$

in particular the function $\tilde{N} = e^{\frac{C}{2}r^2} N(r)$ *is non decreasing and has a limit as* $r \to 0$*, and moreover*

$$\frac{d}{dr} \log(H(r)) = \frac{2}{r}(N(r) - 1).$$

8 Ideas of the proof.

Define the Almgen's quotient as

$$\frac{\frac{1}{r^{N-2}}\int_{B_r(x_0)}\left(|\nabla U|^2-\langle F(x,U),U\rangle\right)}{\frac{1}{r^{N-1}}\int_{\partial B_r}|U|^2}$$

Define the regular and singular parts of the boundary

$$\Sigma_U=\{x\in\Gamma_U\ :\ \lim_{r\to 0}N(x,r)-1=1\}\qquad \Gamma_U\setminus\Sigma_U=\{x\in\Gamma_U\ :\ \lim_{r\to 0}N(x,r)-1>1\}\}$$

- First we wish to apply Federer's reduction principle:
  - Almgren monotonicity formula at nodal points
  - uniform bounds in Hölder spaces $\Longrightarrow$ strong convergence of blow-up sequences. No equation!
  - classification of conic solutions satisfying (WRL)
- Next we analyze the non singular part of the free boundary:
  - flatness at regular points of the boundary
  - clean–up lemma
  - regularity of the nodal set
- Reflection law:
  - (WRL) $\Longrightarrow$ equality of the gradients on the two sides.

9 Conic functions

Lemma 1 *Let $N \geq 2$. Given $\bar{U} = r^\alpha G(\theta) \in Lip_{loc}(\mathbb{R}^N)$ such that $\Delta\bar{U} = 0$ in $\{\bar{U} > 0\}$, and (WRL) holds, then either $\alpha = 1$ or $\alpha \geq 1 + \delta_N$ for some universal constant δ_N depending only on the dimension. Moreover if $\alpha = 1$ then $\Gamma_{\bar{U}}$ is an hyperplane.*

$\bar{U} = r^\alpha(g_1(\theta), \ldots, \gamma_k(\theta))$. Note that for every connected component $A \subseteq \{g_i > 0\} \subset S^{N-1}$ it holds

$$-\Delta_{S^{N-1}} g_i = \lambda g_i \quad \text{in } A, \qquad \text{with } \lambda = \alpha(\alpha + N - 2) \text{ and } \lambda = \lambda_1(A).$$

Lemma 2 *If $\{G > 0\}$ has at least three connected components then there exists an universal constant $\bar{\delta}_N > 0$ such that $\alpha \geq 1 + \bar{\delta}_N$.*

Proof: At least one of the connected components, say C, must have a measure less that one third of the measure of the sphere, and hence $\lambda = \lambda_1(C) \geq \lambda_1(E(\pi/2))$. Moreover it is well know that $\lambda_1(E(\pi/2)) = N - 1$. This implies the existence of $\gamma > 0$ such that $\lambda_1(E(\pi/3)) = N - 1 + \gamma$, and thus $\alpha = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda} - \frac{N-2}{2} \geq 1 + \bar{\delta}_N$ for some $\bar{\delta}_N > 0$. ∎

➡ Use an inductive argument on the dimension, starting with dimension 2.

10 Flatness at the regular points

Let x be a regular point:

Lemma 3 *For any given* $0 < \delta < 1$ *there exists* $R > 0$ *such that for every* $x \in \Gamma^\star \cap \tilde{\Omega} = \Gamma_U \cap \tilde{\Omega}$ *and* $0 < r < R$ *there exists an hyper-plane* $H = H_{x,r}$ *containing* x *such that*

$$d_{\mathcal{H}}(\Gamma_U \cap B_r(x), H \cap B_r(x)) \leq \delta r.$$

Proposition 1 (Local Separation Property) *Given* $x_0 \in \Sigma_U$ *there exists a radius* $R_0 > 0$ *such that* $B_{R_0}(x_0) \cap \Sigma_U = B_{R_0}(x_0) \cap \Gamma_U$ *and* $B_{R_0}(x_0) \setminus \Gamma_U = B_{R_0}(x_0) \cap \{U > 0\}$ *has exactly two connected components* Ω_1, Ω_2*. Moreover, for sufficiently small* $\delta > 0$*, we have that given* $y \in \Gamma_U \cap B_{R_0}(x_0)$ *and* $0 < r < R - |y|$ *there exist a hyper-plane* $H_{y,r}$ *(passing through* y*) and a unitary vector* $\nu_{y,r}$ *(orthogonal to* $H_{y,r}$*) such that*

$$\{x + t\nu_{y,r} \in B_r(y) : \ x \in H_{y,r}, \ t \geq \delta r\} \subset \Omega_1, \qquad \{x - t\nu_{y,r} \in B_r(y) : \ x \in H_{y,r}, \ t \geq \delta r\} \subset \Omega_2.$$

11 The reflection law in action

Lemma 4 (Strong Reflection Law) *Let $u, v \in Lip_{loc}(\mathbb{R}^N)$ be two non zero and non negative functions in $\mathbb{R}^N$ such that $u \cdot v = 0$ and*

$$\begin{cases} -\Delta u = f(x,u) - \lambda \\ -\Delta v = g(x,v) - \mu \end{cases} \qquad \text{in } \mathbb{R}^N$$

for some $\lambda, \mu \in \mathcal{M}_{loc}(\mathbb{R}^N)$, locally non negative Radon measures supported on the common zero set. Suppose moreover that (WRL) holds. Then

$$\boxed{\lambda = \mu}$$

and in particular $\Delta(u - v) = f(x,u) - g(x,v)$ in $\mathbb{R}^N$. Moreover Γ_U is a regular hyper-surface of codimension 1 at regular points, and Then for every Borel set $E \subseteq \mathbb{R}^N$ it holds

$$\lambda(E) = \int_{E \cap \partial\{u>0\}} -\partial_\nu u \, d\sigma = \int_{E \cap \partial\{v>0\}} -\partial_\nu v \, d\sigma = \mu(E)$$

- In the complex or vector valued case, we obtain that $\|\lambda\| = \|\mu\|$. With this and an iterative argument by Caffarelli we deduce the $\mathcal{C}^1$ regularity of the regular part of the boundary.

The main theorem extends to segregated configurations associated with systems of semilinear elliptic equations on Riemannian manifolds, under an appropriate version of the weak reflection law.

We start with a system of semilinear equations involving the Laplace-Beltrami operator on a Riemannian manifold M:

$$-\Delta_M u_i = f(x, u_i) \qquad \text{where } u_i > 0 \ .$$

We define the "energy" $\tilde{E}$ as

$$\tilde{E}(r) = \tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla_M U|^2 dV_M,$$

where $B_r(x_0)$ is the geodesic ball of radius r. Let us choose normal coordinates $\tilde{x}^i$ centered at x_0. By Gauss Lemma we know that, denoting by $\rho = \sum_i (\tilde{x}^i)^2$ and θ^i the radial and angular coordinates, it holds

$$g = d\rho^2 + \rho^2 \sum_{i,j} b_{ij}(\rho, \theta) d\theta^i d\theta^j.$$

Notice that the variation with respect to the euclidean metric is purely tangential. Moreover the Christoffel symbols vanish at the origin.

13 The reflection law using normal coordinates

In normal coordinates, denoting, as usual, $\tilde{g}_{ij} = g(\partial_i, \partial_j)$ the coefficients of the metric, we require that $\tilde{E}$ satisfies the differential equation:

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\rho U)^2 \, d\sigma_M + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \rho \sum_i \left[f_i(x, u_i)\partial_\rho u_i + \frac{1}{\sqrt{\tilde{g}}} \sum_{k,j} \partial_\rho \left(\sqrt{\tilde{g}}\, \tilde{g}^{kj} \right) \partial_k u_i \partial_j u_i \right] dV_M.$$

Here $\tilde{g} = |\det(\tilde{g}_{kj})|$ and $(\tilde{g}^{kj})$ is the inverse of the matrix $(\tilde{g}_{kj})$. This identity is satisfied also in the case of Lipschitz metrics , by any solution u of the semilinear equation

$$-\Delta_M u = f(x, u).$$

14 Some references:

M. Conti, S. Terracini, G. Verzini, A variational problem for the spatial segregation of reaction–diffusion systems, Indiana Univ. Math. J. 54, 3 (2005) 779–815

M. Conti, S. Terracini, G. Verzini, On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae, Calc. Var. Partial Differential Equations 22, 1 (2005) 45–72

L. Caffarelli, F.-H- Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, *J. Amer. Math. Soc.* (2008)

L. A. Caffarelli, A. L. Karakhanyan, and F. Lin, The geometry of solutions to a segregation problem for non-divergence systems, *Journal of Fixed Point Theory and Applications* (2009)

H. Tavares and S. Terracini, Regularity of the nodal set of segregated critical configurations under a weak reflection law, preprint 2010

N. Garofalo and F.-H. Lin Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.* **35** (1986), no. 2, 245–268.

R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili, Critical sets of solutions to elliptic equations. J. Differential Geom. 51, p. 359-373 (1999).

F. Lin, Nodal sets of solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.* **44** (1991), no. 3, 287–308.

15 Asymptotic limits of a system of Gross-Pitaevskii equations

Consider the following system of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \omega_i u_i^3 - \beta u_i \sum_{j\neq i} \beta_{ij} u_j^2 \\ u_i \in H_0^1(\Omega), \ u_i > 0 \text{ in } \Omega. \end{cases} \qquad i = 1, \ldots, h,$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. Such type of systems arises in the theory of Bose-Einstein condensation in multiple spin states. Here we consider $\beta_{ij} = \beta_{ji} \neq 0$ (which gives a variational structure to the problem) and take $\lambda_i, \omega_i \in \mathbb{R}$ and $\beta \in (0, +\infty)$ large. The existence of solutions for β large is still an open problem for some choices of λ_i, ω_i.

One of the many interesting questions about these systems is the asymptotic study of its solutions as $\beta \to +\infty$ (which represents an increasing of the interspecies scattering length) and study of the regularity of the limiting profiles. In a joint paper with Noris, Tavares, T. and Verzini, we have proved:

- $C^{0,\alpha}$– bounds (for all $0 < \alpha < 1$) for any given L^∞–bounded family of solutions $U_\beta = (u_{1,\beta}, \ldots, u_{h,\beta})$ of the system;
- the possible limit configurations $U = \lim_{\beta\to+\infty} U_\beta$ are Lipschitz continuous.

As a byproduct, we have

Theorem 3 *Let U be a limit as $\beta \to +\infty$ of a family $\{U_\beta\}$ of L^∞–bounded solutions of the system. Then the conclusion of Theorem 1 holds.*

All the required assumptions are satisfied for such limiting profiles, with $f_i(x,s) = f_i(s) = \omega_i s^3 - \lambda_i s$, except for the weak reflection law. The procedure to verify it is the following: defining an approximated energy associated with system - which has a variational structure-,

$$E_\beta(r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U_\beta|^2 - \langle F(U_\beta), U_\beta \rangle\right) + \int_{B_r(x_0)} 2\beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2$$

by a direct calculation it holds

$$\begin{aligned} E'_\beta(r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U_\beta)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(u_{i,\beta}) \langle \nabla u_{i,\beta}, x - x_0 \rangle + \\ + \frac{1}{r^{N-1}} \int_{B_r(x_0)} (N-2) \langle F(U_\beta), U_\beta \rangle - \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \langle F(U_\beta), U_\beta \rangle \, d\sigma + \\ + \frac{4-N}{r^{N-1}} \int_{B_r(x_0)} \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 + \int_{\partial B_r(x_0)} \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \, d\sigma. \end{aligned}$$

We know the following facts:

- there holds strong convergence $U_\beta \to U$ in $H^1 \cap C^{0,\alpha}(\Omega)$ for every $0 < \alpha < 1$,
- and $\int_\Omega \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \to 0$.

Hence, as $\beta \to +\infty$, we prove that U satisfies the weak reflection law.

16 Lotka-Volterra competitive interactions with symmetric competition rates

Consider the following Lotka-Volterra model for the competition between h different species.

$$\begin{cases} -\Delta u_i = f_i(u_i) - \beta u_i \sum_{j\neq i} a_{i,j} u_j \text{ in } \Omega, \\ u_i \geq 0 \text{ in } \Omega, \quad u_i = \varphi_i \text{ on } \partial\Omega. \end{cases}$$

with $\Omega \subset \mathbb{R}^N$ a smooth bounded domain and φ_i positive $W^{1,\infty}(\partial\Omega)$–functions with disjoint supports. We focus on the asymptotic study of solutions as $\beta \to +\infty$. It is not difficult to show that all the possible H^1–limits U of a given sequence of solutions $\{U_\beta\}_{\beta>0}$ (as $\beta \to +\infty$) belong to the class

$$\mathcal{S}(\Omega) = \Big\{ (u_1, \ldots, u_h) \in \left(H^1(\Omega)\right)^h : \ u_i \geq 0 \text{ in } \Omega, \ u_i \cdot u_j = 0 \text{ if } i \neq j \text{ and } -\Delta u_i \leq f_i(u_i), \\ -\Delta\Big(u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j\Big) \geq f_i(x, u_i(x)) - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f_j(x, u_j) \Big\}.$$

Theorem 4 *Let $U \in \mathcal{S}$, then if $a_{ij} = a_{ji}$, $\forall i, j$ the conclusion of Theorem 1 holds.*

Asymmetric competition rates

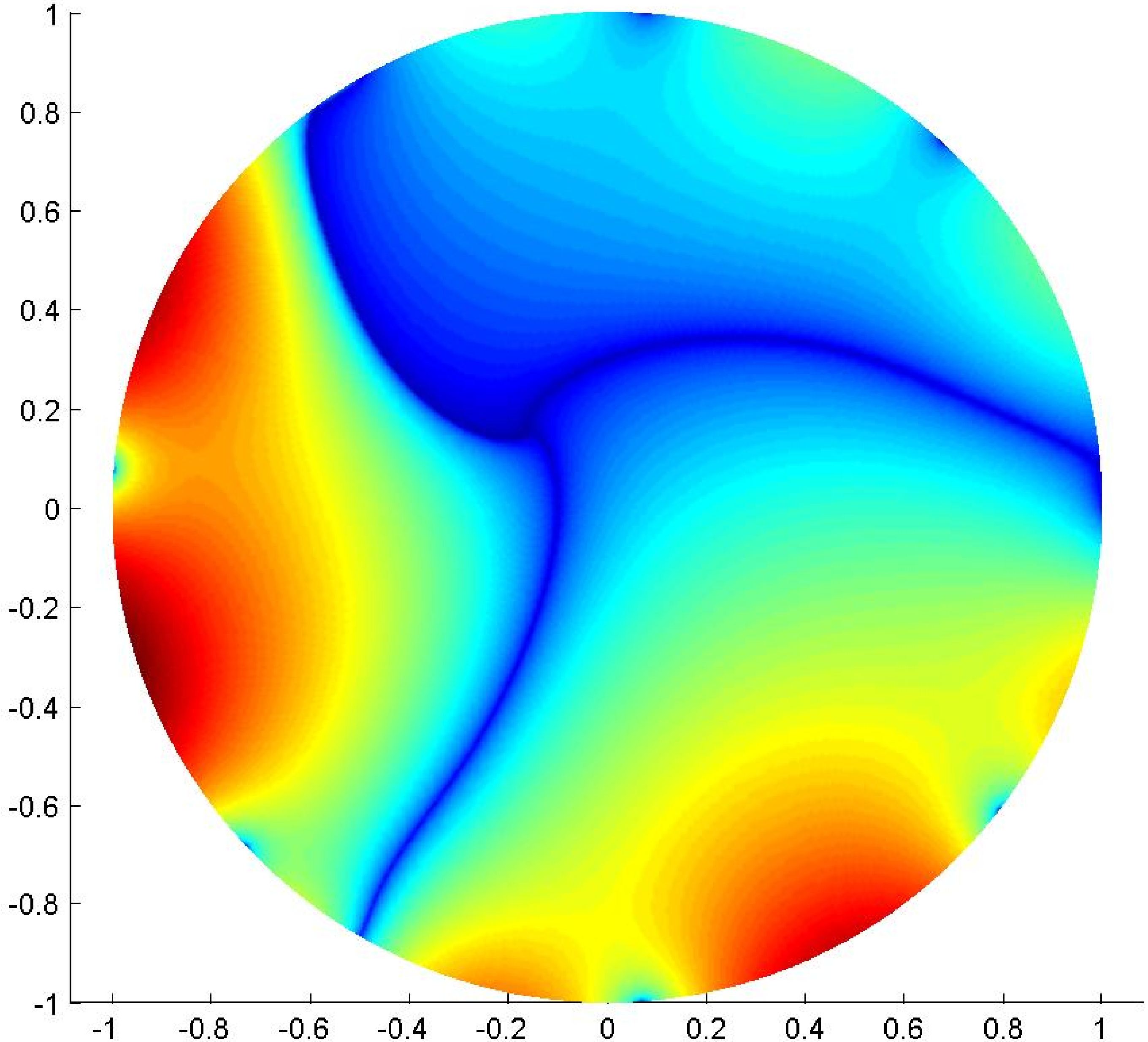
➡ What happens if $a_{ij} \neq a_{ji}$?

➡ What doesn't changes (with respect to the symmetric case):

- lipschitz continuity of the profiles;
- equi–hölderianity w.r. to β;
- equi–lipschitzianity w.r. to β (in the case $h = 2$);
- vanishing of the gradient at multiple points (in dimension $N = 2$).

➡ What changes:

- no monotonicity formula;
- local expansion at multiple points (in dimension $N = 2$).



Next we consider some optimal partition problems involving eigenvalues. For any integer $h \geq 0$, we define the set of h–partitions of Ω as

$$\mathfrak{B}_h = \{(\omega_1, \ldots, \omega_h) : \ \omega_i \text{ measurable },\ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\}.$$

Consider the following optimization problems: for any positive real number $p \geq 1$,

$$\mathfrak{L}_{h,p} := \inf_{\mathfrak{B}_h} \left(\frac{1}{h} \sum_{i=1}^{h} (\lambda_1(\omega_i))^p \right)^{1/p},$$

and, for $p = +\infty$ we find the limiting problem

$$\mathfrak{L}_h := \inf_{\mathfrak{B}_h} \max_{i=1,\ldots,h} (\lambda_1(\omega_i)),$$

where $\lambda_1(\omega)$ denotes the first eigenvalue of $-\Delta$ in $H^1_0(\omega)$ in a generalized sense. We refer to the papers Conti, Verzini, T. and Helffer, Hoffmann-Ostenhof, T., for a more detailed description of these problems.

Our theorem applies to suitable multiples of the eigenfunctions associated with the optimal partition. More precisely, we proved that

- let $p \in [1, +\infty)$ and let $(\omega_1, \ldots, \omega_h) \in \mathfrak{B}_h$ be any minimal partition associated with $\mathfrak{L}_{h,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\omega_i))_i$. Then there exist $a_i > 0$ such that the functions $u_i = a_i \phi_i$ verify in Ω, for every $i = 1, \ldots, h$, the differential inequalities (in the distributional sense):

$$-\Delta u_i \le \lambda_1(\omega_i) u_i \quad \text{and} \quad -\Delta(u_i - \sum_{j \neq i} u_j) \ge \lambda_1(\omega_i) u_i - \sum_{j \neq i} \lambda_1(\omega_i) u_j;$$

and:

- let $(\tilde{\omega}_1, \ldots, \tilde{\omega}_h) \in \mathfrak{B}_h$ be any minimal partition associated with $\mathfrak{L}_h$ and let $(\tilde{\phi}_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\tilde{\omega}_i))_i$. Then there exist $a_i \ge 0$, not all vanishing, such that the functions $\tilde{u}_i = a_i \tilde{\phi}_i$ verify in Ω, for every $i = 1, \ldots, h$, the differential inequalities (in the distributional sense):

$$-\Delta \tilde{u}_i \le \mathfrak{L}_h \tilde{u}_i \quad \text{and} \quad -\Delta(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j) \ge \mathfrak{L}_h(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j).$$

In particular the functions $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_h)$ and $U = (u_1, \ldots, u_h)$ belong to $\mathcal{S}(\Omega)$. As consequence, we have the following result:

Theorem 5 *Let $(\omega_1, \ldots, \omega_h) \in \mathfrak{B}_h$ be any minimal partition and let Γ be the union of the interfaces; then the conclusion of Theorem 1 holds.*

19 Extremality conditions for partitions involving higher eigenvalues

We would like to attack the optimal partition problem for higher eigenvalues ($k \geq 2$):

$$\mathfrak{L} = \min \frac{1}{h} \left(\sum_{i=1}^{h} \lambda_k(\omega_i) \right) .$$

Introduce the penalized functional

$$\mathcal{E}_\beta(u_1, \cdots, u_h) = \int_\Omega \sum_i |\nabla u_i|^2 + \beta \sum_{i \neq j} |u_i|^2 |u_j|^2$$

with constraints

$$\int_\Omega |u_i|^2 = 1 \quad \forall\, i = 1, \cdots, h \, .$$

As $\beta + \infty$, critical points of $\mathcal{E}_\beta$ converge to pairs of segregated eigenfunctions.

Problem: How to define an appropriate critical level for the penalized functional?

Existence of the minimal partition has been proved by Bucur–Buttazzo.

20 More references

M. Conti, S. Terracini, G. Verzini,Asymptotic estimates for the spatial segregation of competitive systems, Adv. Math. 195 (2005), no. 2, 524-560.

M. Conti, S. Terracini, G. Verzini, Uniqueness and Least Energy Property for Strongly Competing Systems, Interfaces and Free Boundaries, 8 (2006), 437-446.

S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin: Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates. *Phys. D* **196**, 341–361 (2004)

E. N. Dancer, Competing species systems with diffusion and large interaction, Rend. Sem. Mat. Fis. Milano 65 (1995), 23-33.

E.N. Dancer, K. Wang and Z. Zhang,*Some New Results in Competing Systems with Many Species*, preprint 2009

E.N. Dancer, K. Wang and Z. Zhang, Dynamics of Strongly Competing Systems with Many Species, preprint 2010

B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition *Comm. Pure Appl. Math.*, **63** (2010), no.3, 267-302.

B. Noris, H. Tavares, S. Terracini and G. Verzini, Convergence of minimax and continuation of critical points for singularly perturbed systems, preprint (2009) Available online at: http://arxiv.org/abs/0910.5317

M. Squassina, S. Zuccher,: Numerical computations for the spatial segregation limit of some 2D competition-diffusion systems, Adv.Math. Sci. Appl. 18 (2008), 83104.

Obviously we have

$$\mathfrak{L}_{k,p} \leq \mathfrak{L}_k \ , \forall k, p.$$

A straightforward consequence of the Fisher–Courant theorem is that

$$\lambda_k \leq \mathfrak{L}_k \ .$$

For one dimensional problems we have equality (Sturm oscillation principle) for every k. On the other hand, in more space dimensions, the k-th eigenfunction may possess less than k nodal domains.

Given k, we denote by L_k the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains ($L_k = +\infty$ if no such an eigenfunction exists).

In general, an easy consequence of the Courant nodal theorem for connected domains is that

$$\lambda_k \leq L_k \ .$$

So we have the inequalities

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \ , \qquad \forall k.$$

➡ Can we characterize the equality cases?

Theorem 6 (Helffer, Hoffman-Ostenhof, T, 2009–2010) *Suppose $\Omega \subset \mathbb{R}^N$ regular. If either $\lambda_k = \mathfrak{L}_k$ or $\mathfrak{L}_k = L_k$ then*

$$\lambda_k = \mathfrak{L}_k = L_k \ .$$

In addition, one can find in the eigenspace associated to λ_k an eigenfunction u_k having exactly k nodal domains.

➡ The k–th eigenfunction has k nodal domains (i.e. is sharp with respect to the Courant nodal Theorem) if and only if the associated nodal k–partition is optimal.

As a consequence, every time we know (for instance for the symmetries of the problem) that the second eigenvalue is degenerate, then the minimal spectral 3-partition has necessarily a nontrivial clustering point.

22 The case of the sphere

We consider the Laplace-Beltrami operator on the two-sphere.

Conjecture 1 (Bishop 1992) *The minimal 3-partition for* $\frac{1}{3}(\sum_{i=1}^{3}\lambda_1(D_i))$ *corresponds to the* $\mathbf{Y}$*-partition, whose boundary is given by the intersection of* $\mathbb{S}^2$ *with the three half-planes defined respectively by* $\phi = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$

The conjecture can be restated as

$$\mathfrak{L}_{3,1}(\mathbb{S}^2) = \frac{15}{4}$$

and also

$$\mathfrak{L}_{3,p}(\mathbb{S}^2) = \mathfrak{L}_{3}(\mathbb{S}^2) = \frac{15}{4}\,, \forall p$$

Bishop's Conjecture was motivated by the analysis of the properties of harmonic functions in conic sets. A reference paper in this context is that by Friedland-Hayman. It is proved there that the optimal two-partition is achieved by the two half spheres.

23 Uniqueness for $\mathfrak{L}_3$ in two dimensions

Theorem 7 (Helffer, Hoffman-Ostenhof, T) *Any minimal spectral 3-partition of $\mathbb{S}^2$ is (up to a rotation) obtained by the **Y**-partition. Hence*

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,.$$

Consider a homogeneous function in $\mathbb{R}^3$ of the form

$$u(x) = r^\alpha g(\theta, \phi)$$

which is harmonic outside its nodal set: $-\Delta u = 0$, $u > 0$ and such that the nodal set divides the sphere in three parts, then

$$\alpha(\alpha + 1) \geq \mathfrak{L}_3(\mathbb{S}^2)\,.$$

Hence our theorem implies that $\alpha \geq 3/2$.

 Ideas of the proof:

- first, minimal partitions on $\mathbb{S}^2$ in three parts exist and share the same properties as for planar domains: regularity and equal angle meeting property. Hence the nodal set is a finite union of arcs.
- because the second eigenvalue of the Laplace-Beltrami operator is singular (has nontrivial multiplicity), then the minimal 3-partition cannot be a nodal partition.
- use Euler's formula and deduce that the nodal line s of a minimal 3-partitions consists exactly two points x_1 and x_2 and three arcs joining these two points.
- use Borsuk (or Ljusternik-Schnirelman) theorem to prove that the nodal set contains a pair of antipodal points.
- the next point is that any minimal 3-partition which contains two antipodal points in its boundary can be lifted to a symmetric 6-partition on the double covering $\mathbb{S}^2_{\mathcal{C}}$.
- finally, the last point is to show that on the double covering a minimal symmetric 6-partition is necessarily the lifting on the double covering of the **Y**-partition.
  - use the knowledge of the spectrum of the Laplace-Beltrami on the double covering and classify all odd and even spectrum.
  - use again the characterization of the eigenvalues whose nodal partition is minimal: this holds if and only the minimal eigenvalue whose nodal partition has k nodal domain is the minimal one,

References

B. Helffer, T. Hoffmann-Ostenhof and S. Terracini, Nodal domains and spectral minimal partitions *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (2009), 101–138.

B. Helffer, T. Hoffmann-Ostenhof and S. Terracini, On Spectral Minimal Partitions: the Case of the Sphere, to appear in Around the Research of Vladimir Mazya III. Analysis and Applications Ari Laptev (Ed.) *International Mathematical Series*, Vol. 13, Springer, 2010

B. Helffer, T. Hoffmann-Ostenhof and S. Terracini, Nodal minimal partitions in dimension 3. Discrete Contin. Dyn. Syst. 28 (2010), no. 2, 617–635.