Lower bounds on Ricci curvature and quantitative behavior of singular sets

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This talk represents joint work with Aaron Naber (who will give a closely related talk on harmonic maps and minimal hypersurfaces). Let Y^n denote the Gromov-Hausdorff limit of a sequence of riemannian manifolds M_i^n with Ricci curvature $\geq -(n-1)$ and $\mathrm{Vol}(B_1(m_i) \geq v > 0$, for all $m_i \in M_i^n$. For all $y \in Y^n$, every tangent cone Y_y is a metric cone. The stratification $\mathcal{S}_0 \subset \cdots \subset S_{n-2}$ off the singular set $\mathcal{S}$ is defined by: $y \in S_k$ if no Y_y splits of a factor $\mathbb{R}^{k+1}$ isometrically. It is known that $\dim \mathcal{S}_k \leq k$ where dim denotes Hausdorff dimension. We give a "quantitative" version of this stratification and corresponding effective bounds in terms of Minkowski content. The argument involves the general method of "quantitative differentiation" and particular tools which exploit the conical structure. In the Einstein case, the results can be combined with known ϵ-regularity theorems, yielding lower bounds on the "curvature radius" $r_{|Rm|}$ off sets of appropriately small volume. By definition, $r_{|Rm|}(y)$ is the largest r such that when $B_r(y)$ is rescaled to unit size then off the rescaled metric the norm curvature is bounded uniformly by 1 on the rescaled ball. In particular, if the M_i^n are Kähler-Einstein and a certain characteristic number is uniformly bounded by C, then for any y the volume of the subset of $B_1(y)$ consisting of points y' with $r_{|Rm|}(y') \leq r$, is bounded by $c(n, v, C)r^4$.