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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Lauren Williams

Talk Title: Introduction to cluster algebras II

Date: 08/23/12 Time: 9:00 am / pm (circle one)

List 6-12 key words for the talk: cluster algebra, positivity, Grassmanian

Please summarize the lecture in 5 or fewer sentances: The lecture explains what a cluster algebra is and provides relevant examples.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Lauren Williams - <u>Cluster algebras II</u>

Let B be an $m \times n$ $(m \geq n)$ integer matrix
We require that the top $n \times n$ part is skew-symmetrizable.
Start with an initial seed

$$\Sigma = \Big(\{ \underbrace{\underbrace{x_1, \ldots, x_n}_{\text{cluster variables}}, \underbrace{x_{n+1}, \ldots, x_m}_{\text{coeff. variables}}}_{\text{extended cluster}} \}, B \Big)$$

Let $C = \{ x_{n+1}, \ldots, x_m \}$ be called the coefficients.
For $1 \leq k \leq n$, get new seed

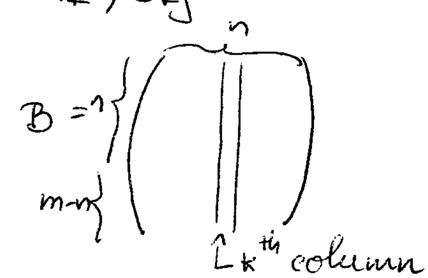
$$\Big(\{ x_1, \ldots, \hat{x}_k, \ldots, x_n, x_k', x_{n+1}, \ldots, x_m \}, \mu_k(B) \Big)$$

where

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} & \text{if } b_{ik} b_{kj} \leq 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

and x_k' is defined by

$$x_k x_k' = \prod_{b_{ik} > 0} x_i^{|b_{ik}|} + \prod_{b_{ik} < 0} x_i^{|b_{ik}|}$$



Start from Σ and apply all sequences of mutations. This produces the set of all cluster variables.
<u>Rk</u>: Each cluster variable is in $\mathbb{Q}(x_1, \ldots, x_m)$
<u>Laurent phenomenon</u>: Each cluster variable is a Laurent polynomial in the initial $\overset{\text{extended}}{\ }$ cluster variables.

<u>Def</u> The <u>cluster algebra</u> $\mathcal{A}(B)$ is the $k[\mathbb{C}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables.

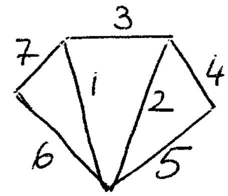
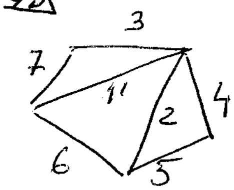
One application of cluster algebras is to find nice (vector space) bases.
<u>Def</u> A <u>cluster monomial</u> is a monomial in the cluster and coefficient variables whose support lies in one extended cluster.

__Conjecture__: The cluster monomials in any cluster alg. are linearly independent.

Back to triangulations:
Any triangulation T leads to a rectangular exchange matrix.
Label diagonals of the $(n+3)$-gon by $1, \dots, n$ and boundary segments by $n+1, \dots, m$ where $m-n = n+3$.
As before $B(T) = (b_{ij})$

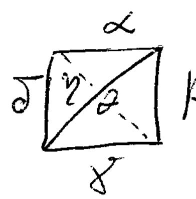
$b_{ij} = \#\{$triangles with sides i and j w/ j following i in clockwise order$\}$

$- \#\{$triangles with sides i and j w/ j following i in counterclockwise order$\}$

__Ex__: $T =$  $\xrightarrow{\text{flip at } 1}$ $T' =$ 

$n=2$

$$B(T) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \\ 4 & 0 & -1 \\ 5 & 0 & 1 \\ 6 & -1 & 0 \\ 7 & 1 & 0 \end{array} \quad \xrightarrow{\mu_1} \quad \mu_1(B(T)) = \begin{array}{cc|c} 1' & 2 & \\ \hline 0 & 1 & 1' \\ -1 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 1 & 5 \\ 1 & -1 & 6 \\ -1 & 0 & 7 \end{array}$$

__Rk__ For each rectangle  $\theta\eta = \alpha\gamma + \beta\delta$

This cluster algebra with coefficients associated to the $(n+3)$-gon can be identified with the coordinate ring of the affine cone over the Grassmanian $Gr_{2,n+3}$.
Call this cluster algebra A_n.

__Def__: The Grassmanian $Gr_{kn}(\mathbb{C}) = \{k\text{-planes in } \mathbb{C}^n\}$

$= \{$full rank $k \times n$ matrices $A\} / A \sim A'$ if their rows span the same subspace$\}$

Given $I \in \binom{[n]}{k}$, the Plücker coordinate $P_I(A)$ = the det of the submatrix of A in columns I.

<u>Ex</u> $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \rightsquigarrow P_{34}(A) = ad - bc$

For $Gr_{2,n}(\mathbb{C})$, the Plücker coordinates are $P_{ij}(A)$ = det of 2×2 submatrix of A in columns i and j.

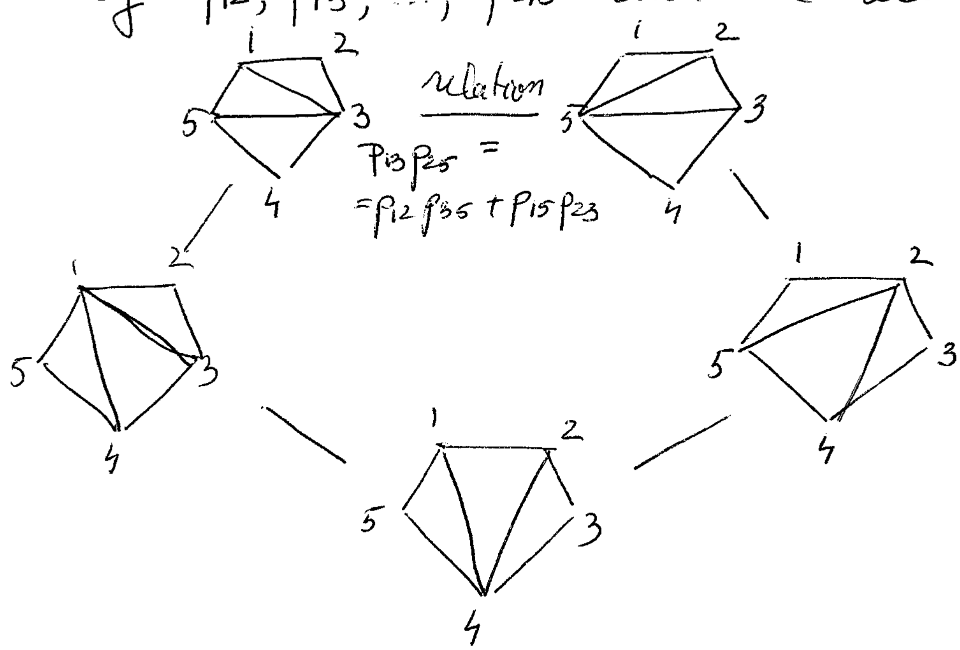
The coordinate ring $\mathbb{C}[Gr_{2n}]$ is generated by the Plücker coordinates P_{ij} subject to relations

$P_{ac}P_{bd} = P_{ab}P_{cd} + P_{ad}P_{bc}$

These are precisely the exchange relations of $\mathcal{A}_{n-3}$.

The exchange graph of $\mathcal{A}_{n-3}$ is the graph whose vertices are triangulations connected by flips.

<u>Ex</u> The coordinate ring of $Gr_{25}(\mathbb{C})$ is generated by $P_{12}, P_{13}, \dots, P_{45}$ and the relations are encoded by



In general, the exchange graph of $\mathcal{A}_n$ is the 1-skeleton of a convex polytope called the associahedron.

For $\mathcal{A}_n$, the cluster monomials are monomials in Plücker coordinates which are supported on a triangulation.

<u>Ex</u>: $P_{14}^3 P_{24} P_{12} P_{45}$

(19th century invariant theory)

<u>Thm</u>: The cluster monomials for $\mathcal{A}_{n-3} \cong \mathbb{C}[Gr_{2,n}]$ form an additive basis for $\mathcal{A}_{n-3}$.

Connections between total positivity and cluster algebras.

<u>Def</u>: The totally positive part $(Gr_{k,n})_{>0}$ of $Gr_{kn}(\mathbb{R})$ is the subset of $Gr_{k,n}(\mathbb{R})$ where all Plücker coordinates are positive.

<u>Question</u>: Given $A \in Gr_{k,n}(\mathbb{R})$, how many Plücker coordinates and which ones do we need to test to determine whether $A \in (Gr_{k,n})_{>0}$?.

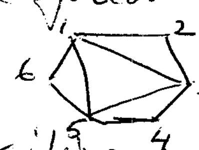
Useful fact: $\mathbb{C}[Gr_{k,p}]$ has the structure of a cluster algebra of rank $(k-1)(p-k-1)$ with n coefficient variables. (This is a theorem by J. Scott)

Rank $(k-1)(p-k-1)$ means each cluster contains $(k-1)(p-k-1)$ cluster variables ~~plus~~ extended p coefficient variables.

<u>Consequence</u>: Any extended cluster (of size $(k-1)(p-k-1)+p$ $= k(p-k)+1$) gives a positivity test.

<u>Ex</u> in the case of $Gr_{2,6}$, if all of the following Plücker coordinates are positive

$\{p_{13}, p_{35}, p_{15}, p_{12}, p_{23}, \ldots, p_{56}, p_{16}\}$



then <u>every</u> Plücker coordinate is positive

$$\text{e.g. } p_{36} = \frac{p_{13}p_{56} + p_{16}p_{35}}{p_{15}}$$

Idea: a space has a coordinate ring with a cluster algebra structure $\Rightarrow$ the space has a natural notion of a "totally positive" part.

Then the (extended) clusters provide positivity tests for membership in the totally positive part.