Hilbert-Kunz multiplicity and Hilbert-Kunz slope

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Let R be a commutative *Noetherian* ring.
E.g. a quotient of a polynomial ring over a field, say

$$R = \frac{k[X_1, \ldots, X_n]}{(f_1, \ldots, f_m)},$$

where $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$.

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Such rings are called *geometric rings* as one can associate a geometric object X_R, called variety, to such a ring R:

$$\begin{aligned} X_R &= \text{the zero set of } \{f_1, \ldots, f_m\} \text{ in } k^n \\ &= \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0, \ \forall \ 1 \leq i \leq n\}. \end{aligned}$$

Moreover a maximal ideal of R, say $\mathbf{m}$ corresponds to a closed point, $x_{\mathbf{m}}$, of X_R.

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This often reduces the work to study the property at the local rings (means rings localized at a point), which could be easier to deal with.

Let R be such a ring of dimension d and $\mathbf{m}$ be a maximal ideal of R. One classical numerical invariant is the *Hilbert-Samuel function* of R at $\mathbf{m}$, namely a function
$HS(R, \mathbf{m}) : \mathbf{N} \to \mathbf{N}$, given by $n \mapsto \ell(R/\mathbf{m}^n)$.

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$HS(R,\mathbf{m}) : \mathbf{N} \to \mathbf{N}$, given by $n \mapsto \ell(R/\mathbf{m}^n)$.
It is a polynomial function, *i.e.*, for $n >> 0$,

$$HS(R,\mathbf{m})(n) = e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d,$$

where $e_0 = e_0(R,\mathbf{m}) \in \mathbf{Z}^+$ is the *classical multiplicity* of R at $\mathbf{m}$.

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For example

1. If X_R is smooth at a point $x_{\mathbf{m}}$ then $e_0(R, \mathbf{m}) = 1$.
   Infact , in general, *if* $(R, \mathbf{m})$ *is an integral domain, then* $e_0(R, \mathbf{m}) = 1$ *if and only if* X_R *is smooth at the point* $x_{\mathbf{m}}$.
2. If R is plane curve with a node at $x_{\mathbf{m}}$ then $e_0(R, \mathbf{m}) = 2$.

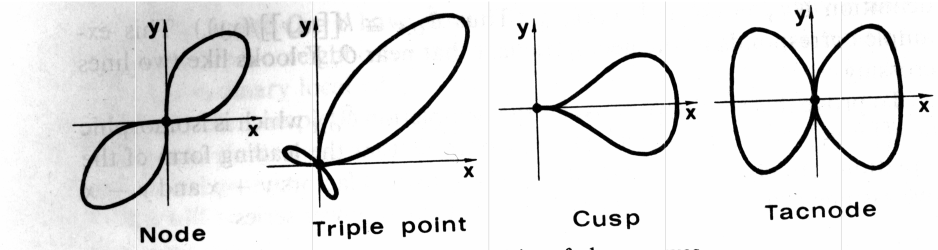

Figure 4. Singularities of plane curves.

$$xy = x^6 + y^6 \text{ (Node)}, \;\; x^3 = y^2 + x^4 + y^4 \text{ (Cusp)},$$

$$x^2 = x^4 + y^4 \text{ (Tacnode)}, \;\; x^2y + xy^2 = x^4 + y^4 \text{ (Triple Point)}$$

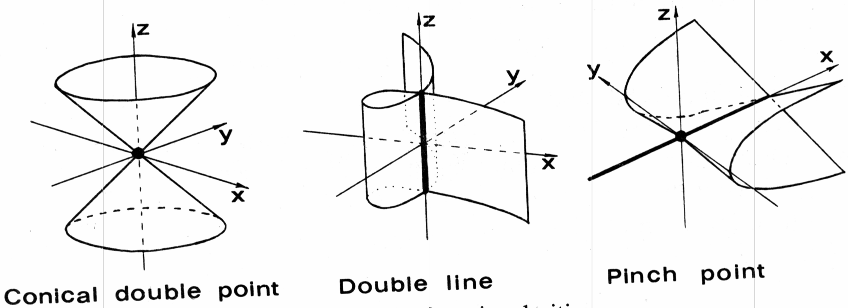

Figure 5. Surface singularities.

$$x^2 + y^2 = z^2 \text{ (Conical Double Point)}, \quad xy = x^3 + y^3 \text{ (Double line)},$$

$$xy^2 = z^2 \text{ (Pinch Point)}$$

In general, larger the multiplicity $e_0(R, \mathbf{m})$, more singular is the variety at the point $x_{\mathbf{m}}$.
Moreover e_0 is a well behaved invariant in the sense,

1. it does not change after taking a general hyperplane section, and
2. remains constant in a flat family.
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2. remains constant in a flat family.
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Now consider a 'char p' invariant of rings which relates to 'char p features of the underlying ring.

Definition: For a commutative local ring, $(R, \mathbf{m})$ of characteristic $p > 0$, we define *Hilbert-Kunz function* $HK(R, \mathbf{m}) : \mathbf{N} \to \mathbf{N}$, as

$$HK(R, \mathbf{m})(p^n) = \ell(R/\mathbf{m}^{(p^n)}),$$

where

$$\begin{aligned} \mathbf{m}^{(p^n)} &= \text{ideal generated by}\{x^{p^n} | x \in \mathbf{m}\} \\ &= F^n(\mathbf{m})R, \end{aligned}$$

where $F^n : R \to R$ is the n-th iterated Frobenius map, given by, $x \mapsto x^{p^n}$.

Monsky (1980's) proved:

$$HK(R,\mathbf{m})(q) = e_{HK}(R,\mathbf{m})q^d + O(q^{d-1}), \text{ where } q = p^n$$

where $e_{HK}(R,\mathbf{m}) \in \mathbf{R}^+$ is the *Hilbert-Kumz multiplicity*, and $O(q^{d-1})$ is a function of q of order $\leq d-1$.

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One can easily see that

$$\frac{1}{d!}e(R, \mathbf{m}) \leq e_{HK}(R, \mathbf{m}) \leq e(R, \mathbf{m}).$$

Theorem

(Monsky) If $\dim\ R = 1$,

$$HK(R)(q) = e_0(R, \mathbf{m})q^n + \Delta_n,$$

$q = p^n$, where Δ_n is a periodic function of n, for $n >> 0$.

Open question (Monsky 1980's): *Is $e_{HK}(R)$ a rational number?*

We recall some examples for which $e_{HK}(R)$ or $HK(R)$ has been computed, by various people K. Pardue, R.Buchweitz-C.Chen, P.Monsky, C.Han-Monsky, A. Conca, Eto, W. Bruns, Watanabe-Yoshida etc.

1. $R =$ a polynomial ring over a field,
2. $R = k[X, Y, Z]/(f)$ a plane curve, Then
    1. R a nodal plane curve,
    2. R an elliptic plane curve and char $k \neq 2$, if R an elliptic plane curve and char $k = 2$,
3. diagonal hypersurfaces,
4. monomial ideals and binomial hypersurfaces,
5. monoid rings, toric ring,
6. More recently, trinomial plane curves,
7. (N.Fakhruddin, V.T.) R a homogeneous cordinate ring of
    1. an elliptic curve embedded via any line bundle of degree ≥ 3, or
    2. a full flag variety embedded by an anticanonical line bundle, Infact here we have

$$HK(q) = e_{HK}q^d + C_1(n)q^{d-1} + \cdots + C_d(n),$$

where $q = p^n$ and $C_i(n)$ are periodic function of n.

We note that the above examples (except the last one) are hypersurfaces of special types or monomial rings. Hence one is able to use combinatorial techniques, grobner bases etc.

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For the last example, inspite of the fact that rings are given by arbitrary large number of equations or of large dimension, the clean computations were possible due to

1. Atiyah's classification of vector bundles for elliptic curves, and
2. the result of Anderson-Haboush: $F_*(\mathcal{L}(p-1)\rho)$ is a trivial bundle for G/B.

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(as shown by the above examples) unlike Hilbert-Samuel multiplicity, the HK multiplicty does *not* remain constant,

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2. going to a flat deformation.

Why is e_{HK} interesting?

Main reason: $e_{HK}(R)$ is a more subtle invariant than $e(R)$ and it reveals more information about the char p features of the ring R.

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1. If R s a domain then $e_{HK}(R, \mathbf{m}) = 1$ if and only if R is smooth at $\mathbf{m}$. (Watanabe-Yoshida)
2. $e_{HK}(R) < 1 + (1/d!)$, where d is the dimension of R, implies that R is F-rational. (M. Blickle and F. Enescu)
   We recall that F-rational is a substitute for rational singularity in char p, as the problem of existence of a resolution of singularity, for the varieties in char p, is still open.

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3. Conjecture (Watanabe-Yoshida): For every nonregular ring R of dimension d and of char p,
   1. $e_{HK}(R) \geq e_{HK}(A_{p,d})$, where $A_{p,d}$ is a quadratic d-dimesional hypersurface in char p,
   $$A_{p,d} = \bar{\mathbf{F}}_p[X_0, \ldots, X_d]/(X_0^2 + \cdots + X_d^2).$$
   2. if equality holds then $R \cong A_{p,d}$ analytically (upto base change by a field).

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Let $R = \oplus_{m\geq 0} R_m$ be a standard graded ring of dimension 2 over a field of characateristic $p > 0$.

Let $\mathbf{m} = \oplus_{m\geq 1} R_m$

Then $X = Proj\ R$ is a projective curve.

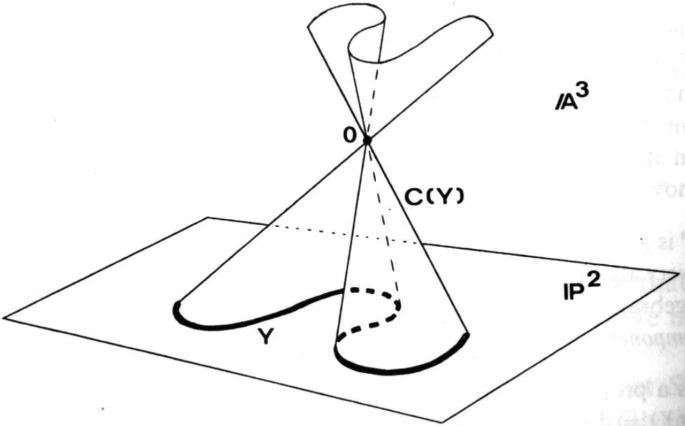

Figure 1. The cone over a curve in $\mathbf{P}^2$.

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Now

$$HK(R, \mathbf{m}) = \ell\left(\frac{R}{\mathbf{m}^{(q)}}\right) = \sum_{m\geq 0} \ell\left(\frac{R_{m+q}}{\text{Im }(R_1^{(q)} \otimes R_m)}\right).$$

Consider the canonical short exact sequence

$$0 \to V \to H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \to 0.$$

Note that V is a vector bundle on X, (i.e., if $\mathcal{O}_X$ is the sheaf of rings then V is a sheaf of free modules on $\mathcal{O}_X$).
Let $F^s : X \to X$ be the s-th iterated Frobenius map.
Then

$$0 \to H^0(X, F^{s*}(V)(m)) \to R_1^{(q)} \otimes R_m \to R_{m+q} \to H^1(X, F^{s*}(V)(m)) \to 0.$$

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Thus the computation of e_{HK} is reduced to the computation of the cohomologies of a vector bundle whose rank and degree we know.

Definition A vector bundle W on X is *semistable* if for any subbundle $W' \subset W$, we have $\mu(W') \leq \mu(W)$, where

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Lemma For a semistable bundle W of rank r and curve of genus g, we have

1. $h^0(X, W(m)) = 0$, if $\deg W(m) < 0$ and
2. $h^1(X, W(m)) = 0$, if $\deg W(m) > r(2g-2)$,
3. $h^0(X, W(m)) \leq rg$, if $0 \leq \deg W(m) \leq r(2g-2)$.

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So we could have carried out the computation in terms of the known invariants like degree and rank of V, if V and $F^{s*}(V)$ were semistable for $s \geq 0$.

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every bundle V has a *Harder-Narasimhan filtration*,

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_t \subset F_{t+1} = V$$

such that

1. F_{i+1}/F_i is semistable for every i,
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Let us define

$$a_{HK}(V) := \sum_i \mu_i(V)^2 r_i(V).$$

But a semistable bundle need not be strongly semistable, that is,

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However, by a (not so old) result of A.Langer,
given a vector bundle V, if $s >> 0$ then the HN filtration of $F^{s*}V$ is strongly semistable.
In particular, for the HN filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = F^{s*}V,$$

each E_{i+1}/E_i is strongly semistable.

This gives us a well defined notion called *HK slope* of V, as

$$\mu_{HK}(V) := \frac{1}{p^s}\sum_i \mu_i(F^{s*}(V))^2 r_i(F^{s*}(V)), \quad \text{where} \quad r_i = \text{rank}\frac{E_i}{E_{i-1}} \quad .$$

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Theorem

(Brenner, V.T.) If R is a standard graded two dimensional over a field of char $p > 0$, then

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In particular for a standard graded 2 dimensional ring e_{HK} is a rational number.
This generalizes the result of Monsky, namely if dim $R = 1$ then $e_{HK}(R, \mathbf{m}) = e_0(R, \mathbf{m})$,
Question is still open for nongraded 2 dimensional rings.

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Infact e_{HK} gives information about the Frobenius semistabiltiy behaviour of V.

Theorem

(V.T.) For R as above,

$$e_{HK}(R) \geq \frac{\deg\ X}{2}\left[1 + \frac{1}{(embdim(R)) - 1}\right].$$

Moreover equality holds if and only if V is strongly semistable.

In the case of plane curves e_{HK} gives a numerical characterization of the Frobenius semistablity behaviour of the syzygy bundle.

Theorem

(V.T.) Let X be a nonsingular plane curve of degree $d > 1$. Let V be the syzygy bundle given by the canonical map

$$0 \to V \to H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \to 0.$$

Then one of the following holds:

1. *V is strongly semistable. In this case $e_{HK}(X) = 3d/4$.*
2. *V is not semistable. Then $e_{HK}(X) = \frac{3d}{4} + \frac{l^2}{4d}$, where $0 < l < d$ and l is an integer congruent to d (mod 2).*
3. *V is semistable but not strongly semistable. Let $s \geq 1$ be the number such that $F^{(s-1)*}V$ is semistable and $F^{s*}V$ is not semistable. Then*

$$e_{HK}(X) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}},$$

where l is an integer congruent to pd (mod 2) with $0 < l \leq 2g - 2$, so that in particular $0 < l \leq d(d-3)$.

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Here we crucially use a Shepherd-Barron and X.Sun's inequality:

If X is a nonsingular projective curve of genus g and if V is a semistable vector bundle on X of rank r such that F^*V is not semistable then

$$0 < \mu_{\max}(F^*V) - \mu_{\min}(F^*V) \leq (2g-2)(r-1).$$

Consider the example,

$$R_p = k[X, Y, Z]/(x^4 + y^4 + z^4), \text{ where char } k = p.$$

$e_{HK}(R_p)$ is computed by Han-Monsky,
Now applying our numerical characterization to this example and its syzygy bundle V_p, we have

1. V_p is strongly semistable if $p \equiv \pm 1(8)$, or char k is zero.
2. V_p is semistable but $F^* V_p$ is not semistable if $p \equiv \pm - 3(8)$.

Conclusion: The semistability of Frobenius pull backs does not behave well under 'reduction mod p'. Though semistability itself is the open property.

However, we can say HN filtraion of V_p behaves well as p is large in the following sense:

Theorem

(V.T.) Let V be a vector bundle of rank r, on a nonsingular curve X of genus g, with the HN filtration as

$$E_1 \subset E_2 \subset \cdots \subset E_l \subset V.$$

Assume that $\mathrm{char}\, k = p > 4(g-1)r^3$. Then,

$$F^*E_1 \subset F^*E_2 \subset \cdots \subset F^*E_l \subset F^*V$$

is a subfiltration of the HN filtration of F^*V, that is, if

$$0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_{l_1+1} = F^*V$$

is the HN filtration of F^*V then for every $1 \leq i \leq l$ there exists $1 \leq j_i \leq l_1$ such that $F^*E_i = \tilde{E}_{j_i}$.

Theorem

(V.T.) Let $f : X_A \to \mathrm{Spec}\, A$ *be a family of smooth projective curves, where* A *is a finitely generated* $\mathbb{Z}$*-algebra. Let* $\mathcal{O}_{X_A}(1)$ *be a* f*-very ample sheaf on* X_A*, then*

1. $$\lim_{s \mapsto s_0} \mu_{HK}(V_s) = a_{HK}(V_{s_0}),$$

   where s_0 *is the generic point of* $\mathrm{Spec}\, A$ *and* s *is the closed point of* $\mathrm{Spec}\, A$*. In particular*

2. $$\lim_{s \mapsto s_0} e_{HK}(X_s) = \frac{\deg\ X_{s_0}}{2}(a_{HK}(V_{s_0}) - \mathrm{embdim}\ \mathcal{O}_{X_{s_0}}).$$

In particular,
if R is a standard graded 2-dimensional ring over a field of char 0 and I a graded maximal ideal and A is a f.g. $\mathbb{Z}$ algebra then for the pair (R, I), and any choice of a spread of (A, R_A, I_A), we can define

$$e_{HK}(R, I) = \lim_{s \mapsto s_0} e_{HK}(R_s, I_s).$$

Statement (1) of the above theorem holds for families of higher dimensional projective varieties.

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$$\lim_{s \mapsto s_0} \mu_{HK}(V_s) = a_{HK}(V_{s_0}).$$

Note that for $p >> 0$ (in terms of the well behaved invariants of the bundle and the ambient variety),

1. $a_{HK}(V_p)$ is a constant, hence can write as $a_{HK}(V)$ (as the semistability property is an 'open' property), and
2. $\mu_{HK}(V_p) = a_{HK}(V)$ if and only if the HN filtration of V_p is the strong HN filtration.

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2. $\mu_{HK}(V_p) = a_{HK}(V)$ if and only if the HN filtration of V_p is the strong HN filtration.

In particular though Frobenius semistability does not behave well under *reduction mod p*, we have

$$\lim_{p \mapsto \infty} \mu_{HK}(V_p) = a_{HK}(V).$$

Again for this, first we generalize the Shepherd-Barron inequality to higher dimension and then the above theroem the higher dimension, stating that

the HN filtration of a Frobenius pull back of V is a refinement of a Frobenius pull back of the HN fitration of V.

One can use the examples of Raynaud and Monsky to say that this statement is *not true* for smaller p compare to genus of X or rank of V.

What about e_{HK} in higher dimensions?

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Remark: (Monsky's conjecture)

1. If $R = \frac{\mathbf{Z}}{2}[X, Y, Z, u, v]/(H + uv)$, then
$$e_{HK}(R) = \frac{4}{3} + \frac{5}{14\sqrt{7}}$$
2. If $R = k[X_1, \ldots, X_9]/(f)$, then $e_{HK}(R) \in \mathbf{R}^+$ is transcendental.

Huneke-Monsky-Macdormatt proved (under some mild conditions) that

$$HK(R, \mathbf{m})(q) = e_{HK}(R, \mathbf{m})q^d + \beta(R)q^{d-1} + f(n),$$

where $f(n) = O(q^{d-2})$,. Moreover there exists cases where $f(n) \neq \nu(R)q^{d-2} + O(q^{d-3})$.