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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Gordana Todorov

Talk Title: Relations between Cluster Algebras & Cluster Categories

Date: 08/23/12 Time: 4:00 am / pm (circle one)

List 6-12 key words for the talk: cluster category, cluster algebra, indecomposable objects, cluster tilting objects

Please summarize the lecture in 5 or fewer sentances: Cluster categories were introduced in order to give categorical interpretation to the combinatorics of cluster algebras. The talk illustrates the relations between the two.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>Computer Presentations</u>: Obtain a copy of their presentation
  - <u>Overhead</u>: Obtain a copy or use the originals and scan them
  - <u>Blackboard</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>Handouts</u>: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

Gordana Todorov - Relation between cluster algebra and cluster categories

Motivation - give module theoretical interpretation to the notions and combinatorics of cluster algebras

Correspondence (assumptions: no coefficients, acyclic)

$\{$ cluster variables $x_i \}$ $\xleftrightarrow{1-1}$ $\{$ indecomposable modules (objects) with no self-extensions $\}$

$\{$ clusters $\underline{x} = \{x_1, \dots, x_n\} \}$ $\longleftrightarrow$ $\{$ cluster tilting objects $\}$ $T = T_1 \oplus \dots \oplus T_n$

$\{$ cluster seeds $(\underline{x}, B) \}$ $\longleftrightarrow$ $\{$ cluster tilting seeds $\}$ $(T, End(T))$

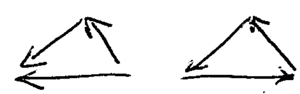
Need
- quiver representations or modules over path algebra
- AR (Auslander - Reiten) quiver, translation
- $\mathcal{D}^b(kQ)$ derived category
- orbit category
- $\underline{\underline{x}}$ - approximation

Quivers $Q = (Q_0, Q_1)$; $k = \bar{k}$

vertices $\to Q_0$, $Q_1 \leftarrow$ ~~arrows~~ arrows

$\{1, 2, \dots, n\}$

assume Q has no oriented cycles



OK not OK

Quiver representations

$$V = \begin{cases} V_i = k \text{ vector space} & \forall i \in Q_0 \text{ vertex} \\ V_i \xleftarrow{V_a} V_j \quad k\text{-linear map} & \forall\, i \leftarrow j \in Q_1 \text{ arrow} \end{cases}$$

a morphism $V \longrightarrow W$ is a collection of morphisms

$$\begin{tikzcd} V_i \arrow[d, "f_i"'] & V_j \arrow[l, "V_a"'] \arrow[d, "f_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

such that all such squares commute

Ex $1 \xleftarrow{a} 2 \xleftarrow{b} 3$

simple modules

$S_1 : k \longleftarrow 0 \longleftarrow 0$

$S_2 : 0 \longleftarrow k \longleftarrow 0$

$S_3 : \longleftarrow \longleftarrow k$

projective modules

$P_1 = S_1$

$P_2 : k = k \longleftarrow 0$

$P_3 \;\; k = k = k$

injective

$I_2 : 0 \longleftarrow k \longleftarrow k$

AR quiver:

$$\begin{tikzcd}
 & & P_3 \arrow[dr, two heads] & & \\
 & P_2 \arrow[ur, hook] \arrow[dr, two heads] & & I_2 \arrow[dr] & \\
P_1 \arrow[ur, hook] & & S_2 \arrow[ur, hook] & & S_3
\end{tikzcd}$$

Information contained above
- indecomposable modules
- irreducible maps (do not factor)
- AR sequences (almost split sequences)

$0 \to P_1 \to P_2 \to S_2 \to 0$

$0 \to P_2 \to P_3 \oplus S_2 \to I_2 \to 0$

$0 \to S_2 \to I_2 \longrightarrow S_3 \to 0$

- AR translation τ, τ^{-1}

$S_2 = \tau^{-1} P_1$

$S_3 = \tau^{-1} P_2$

$I_2 = \tau^{-1} P_2$

Rk $\operatorname{rep}(Q,k) \simeq \operatorname{mod}(kQ)$

kQ is hereditary

$\operatorname{Ext}^i(\ ,\ ) = 0, \ i \geq 2$

$D^b(kQ)$ = derived category of bounded complexes

- easy to describe for hereditary

Every indecomposable object in $D^b(kQ)$ is isomorphic to a stalk complex; i.e. shift of a module $M[i]$

Cluster category

$\underline{\underline{C}} = D^b(kQ)/F$ orbit category where $F = \tau^{-1} \circ [1]$

Fundamental domain: $\operatorname{mod} kQ \cup \{P_i[1]\}$

Fundamental domain:

$$\begin{tikzcd}[column sep=small]
 & & P_3 \arrow[dr] & & P_1[1] \arrow[dr, dash] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur, dashed, dash, "\mathrm{Ext}^1(I_2,P_1)"] \arrow[dr] & & P_2[1] \arrow[dr, dash] & \\
P_1 \arrow[ur] & & S_2 \arrow[ur] & & S_3 \arrow[ur, dash] & & P_3[1]
\end{tikzcd}$$

Cluster tilting objects

$$\mathrm{Ext}^1_{\underline{\underline{C}}}(A, B) = \mathrm{Hom}_{\underline{\underline{C}}}(A, B[1])$$

T is a cluster tilting object if:

1. $\mathrm{Ext}^1_{\underline{\underline{C}}}(T, T) = 0$

2. $T = T_1 \oplus \dots \oplus T_n$, T_i nonisomorphic indecomposables

E.g:

$$\begin{tikzcd}[column sep=small]
 & P_1 \oplus S_3 \oplus P_3 \arrow[d, dash, "\mu_2"] & \\
 & P_1 \oplus P_2 \oplus P_3 \arrow[dl, dash, "\mu_1"'] \arrow[dr, dash, "\mu_3"] & \\
S_2 \oplus P_2 \oplus P_3 & & P_1 \oplus P_2 \oplus P_3[1]
\end{tikzcd}$$

Cluster tilting mutation

$$T = T_1 \oplus \dots \oplus T_i \oplus \dots \oplus T_n$$

$$T' = T_1 \oplus \dots \oplus T_i' \oplus \dots \oplus T_n = \mu_i(T)$$

Def $\underline{\underline{X}} \subset \underline{\underline{C}}$ subcategory

$A \in \underline{\underline{C}}$ then $\underline{\underline{X}}$ - minimal approximation of A is a map $A \xrightarrow{f_A} X_A$

Such that any map $A \xrightarrow{\alpha} X$ factors as

$$\begin{tikzcd}
A \arrow[r, "f_A"] \arrow[dr, "\alpha"'] & X \arrow[d, dashed] \\
 & X
\end{tikzcd}$$

Mutation $\mu_i(T)$

$T_i \longrightarrow X \longrightarrow T_i'$ complete triangle

add (T/T_i)

approx.

$T_i \xhookrightarrow{f} X \longrightarrow \operatorname{coker} f$

E.g: $P_1 \longrightarrow P_2$ is a minimal approximation in add $P_2 \oplus P_3$

$P_1 \hookrightarrow P_2 \longrightarrow S_2 = \operatorname{coker} \Rightarrow S_2$ replaces P_i

$P_1 \oplus P_2 \oplus P_3 \xrightarrow{\mu_1} S_2 \oplus P_2 \oplus P_3$

$P_3 \longrightarrow 0 \longrightarrow P_3[1]$

$P_1 \oplus P_2 \oplus P_3 \xrightarrow{\mu_3} P_1 \oplus P_2 \oplus P_3[1]$