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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Konstanze Rietsch

Talk Title: On Mirror Symmetry for Grassmanians

Date: ____/____/____ Time: ____:____ am / pm (circle one)

List 6-12 key words for the talk:

Please summarize the lecture in 5 or fewer sentances: The talk reports on work of Marsh and Rietsch which uses the cluster algebra structure of the coordinate ring of a Grassmanian for studying mirror symmetry.

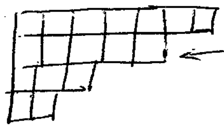
CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Konstanze Rietsch — On mirror symmetry for Grassmannians

$\mathbb{C}[x_1, \dots, x_m]^{S_m}$ = the ring of symmetric polynomials
$e_1, \dots, e_m$ = elementary symmetric polynomials
A basis of the ring of symmetric polynomials is given by the Schur polynomials S_λ, indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ with at most m parts


← λ_1 boxes
← λ_2 boxes
} Young diagram

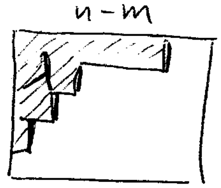
e.g. $e_1 = S_{\square}$
$e_2 = S_{\substack{\square\\\square}}$

Pieri rule: $S_{\square} S_\lambda = \sum_{\substack{\text{all } \lambda^{\square} = \text{partition} \\ \text{obtained by} \\ \text{adding } \square \text{ to } \lambda}} S_{\lambda^{\square}}$

e.g.: $m=2$ $\quad S_{\square} S_{\substack{\square\\\square}} = (x_1+x_2)x_1x_2 = x_1^2x_2 + x_2^2x_1 = S_{\substack{\square\square\\\square\square}}$

$m=3$ $\quad S_{\square} S_{\substack{\square\\\square}} = S_{\substack{\square\square\\\square}} + S_{\substack{\square\\\square\\\square}}$

Recall: $Gr_m(\mathbb{C}^n) = \{ V \subset \mathbb{C}^n \mid \dim V = m \}$
$H^*(Gr_m(\mathbb{C}^n)) = \mathbb{C}[x_1, \dots, x_m]^{S_m} / (S_\lambda, \text{ where } \lambda_1 > n-m)$

i.e.
$n-m$
m

set to zero
$h_k = S_{\underbrace{\square \cdots \square}_{k}}$, $k > n-m$

$\lambda \leftrightarrow I_\lambda = \{ i_1, \dots, i_m \}$
$\uparrow$ a subset of $\{1, \dots, m\}$ indicating where the vertical steps in the path occur

$B^+ = \begin{pmatrix} * & & \\ & \ddots & \\ & & \end{pmatrix}$ $\quad \lambda \rightsquigarrow \overline{B^+ \cdot p_\lambda} = X_\lambda$ Schubert variety

(Small) quantum cohomology ring $qH^*(Gr_m(\mathbb{C}^n))$

$$qH^*(Gr_m(\mathbb{C}^n)) = \mathbb{C}[x_1,\dots,x_m; q]^{S_m}\Big/(h_{n-m+1},\dots,h_n+(-1)^m q)$$

Let σ^λ denote the quantum Schubert class associated to X_λ.

Write $*_q$ for cup product.

$$\sigma_\square *_q \sigma_\lambda = \sum_{\lambda^\square \subset \boxed{m}^{\,n-m}} \sigma^{\lambda^\square} + \sum_{\nu} q\, \sigma^\nu$$

ν is obtained from λ by removing $n-1$ boxes at least 1 in each row

Toy example for B-model

$X = \mathbb{CP}^1 = Gr_1(\mathbb{C}^2)$

$H^*(\mathbb{CP}^1) = \mathbb{C}[h]/(h^2)$

$qH^*(\mathbb{CP}^1) = \mathbb{C}[h,q]/(h^2-q)$

$$\check{X} = \mathbb{C}^* \xrightarrow{W_q} \mathbb{C} \quad \leftarrow \text{superpotential}$$

$$z \longmapsto z + \frac{q}{z}$$

also interested in $W_t = W_{e^t}$, $q = e^t$

Fix $q \neq 0$, $z \in \mathbb{C}$

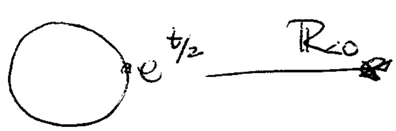
$$\mathbb{C}[\check{X}]/(\partial W_q) = \mathbb{C}[z, z^{-1}]\Big/\left(1-\frac{q}{z^2}\right) = \mathbb{C}[z]/(z^2-q)$$

Here W_t has 2 critical pts $z = \pm e^{\frac{t}{2}}$

$H_1^{\mathrm{rod.}}(\check{X}, \mathrm{Re}(\frac{1}{\hbar} W_t) \to -\infty) \leftarrow$ defines a vector bundle over $\mathbb{C}$ with Gauss-Manin connection

e.g.

basis



Back to the A-model:

$$\begin{array}{c} H^*(\mathbb{CP}^1)\times\mathbb{C} \\ \downarrow pr_2 \\ \mathbb{C}\end{array}$$

Dubrovin - Givental connection

$$\nabla = d - \frac{1}{\hbar}\, h *_{e^*} -$$

So flat sections are maps $s : \mathbb{C}\to H^*(\mathbb{CP}^1)$ which satisfy $\frac{d}{dt} S = \frac{1}{\hbar}\, h *_{e^*} S(t)$.

$$H^*(\mathbb{CP}^1) \longrightarrow H_1^{rd}(\check{X}_t, -)$$

$$1 \longmapsto \left(\oint e^{-\frac{1}{\hbar}W_t}\frac{dz}{z}\right)[\mathbb{R}_{<0}] + \left(\int_{-\infty}^{0} e^{-\frac{1}{\hbar}W_t}\frac{dz}{z}\right)[0]$$

$$h \longmapsto \left(\text{—}\,''\,\text{—}\; dz\right)[\mathbb{R}_{<0}] + \left(\text{—}\,''\,\text{—}\; dz\right)[0]$$

$$H_1^{rd}(\check{X}\ldots) = H_1^{dR}\left(\check{X}, d - \frac{1}{\hbar}dW\right)$$

dual space to the $\mathbb{C}^n$ before

$\check{X}\subset Gr_{n-m}\left((\mathbb{C}^n)^*\right)$ $\qquad n-m=k$

Recall $X = Gr_m(\mathbb{C}^n)$

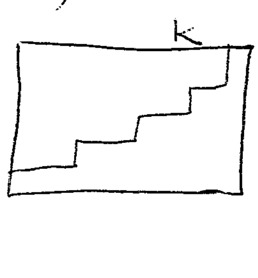
k-subsets of $\{1,\ldots,n\}$ $\leftrightarrow$ Plücker coordinates

$\check{X}$: $\Delta_{\{1,\ldots,k\}} =: \Delta_{\mu_k} \neq 0$

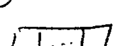
$\Delta_{\{2,\ldots,k+1\}} =: \Delta_{\mu_{k+1}} \neq 0$

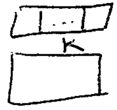
$\vdots$

$\Delta_{\{n,1,\ldots,k-1\}} =: \Delta_{\mu_{k-1}} \neq 0$

partition $\lambda \subset$ $n-k$  $\longleftrightarrow I_\lambda \subset \{1,\ldots,n\}$

$\mu_K = \emptyset$

$\mu_{Kn} =$  ,

$\mu_n =$  $n-k$

Define $\mu_n^\square =$  otherwise $\mu_j^\square = \mu_j$ with box added

$$W_t = \sum_{j \neq n} \frac{\Delta_{\mu_j^\square}}{\Delta_{\mu_j}} + e^* \frac{\Delta_{\mu_n^\square}}{\Delta_{\mu_n}}$$

Thm (joint with R. Marsh)

$$H^*(X) \longrightarrow H_N^{dR}(\check{X}, \nabla = d - \tfrac{1}{\hbar} dW)$$

$$\nabla^\lambda \longmapsto [\Delta_\lambda \, \omega]$$

where $\omega = \bigwedge_{\Delta_i \in C} \frac{d\Delta_i}{\Delta_i}$

C some cluster