Analytic Isomorphisms of Artin K-algebras

Maria Evelina Rossi

University of Genova, Italy
Department of Mathematics

http://www.dima.unige.it/ rossim/Berkeley.pdf

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Formal Power series ring

Denote

$$R = K[[x_1, \ldots, x_n]]$$

where $K = \overline{K}$ and char $K = 0$.

We recall that a K-algebra automorphism of R acts as **substitution**

$$\varphi : \quad R \to R$$

$$x_j \rightsquigarrow y_j = f_j(x_1, \ldots, x_n)$$

such that $\mathfrak{m}_R = (x_1, \ldots, x_n) = (y_1, \ldots, y_n)$.

$$\varphi \in Aut(R) \quad \Longleftrightarrow \quad J_\varphi(0) \neq 0 \text{ (Jacobian condition)}$$

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Given I, J ideals of R, there exists an analytic isomorphism

$$R/I \xrightarrow{\sim} R/J \iff \exists\, \varphi \in Aut(R) \text{ s.t. } \varphi(I) = J.$$

We will write

$$I \sim J.$$

Example

Consider $R = \mathbb{C}[[x, y]]$ then

$$I = (y^2 - x^3, x^3 y) \sim J = (y^2 - x^2 y - x^3, x^3 y)$$

$$\varphi : R \to R$$

$$x \rightsquigarrow 9x + y$$

$$y \rightsquigarrow -27y + xy + 9x^2$$

It is easy to see that $\varphi(I) \subseteq J$ and we conclude because they have the same Hilbert function $\{1, 2, 2, 2, 2, 1\}$.

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Notations

From now on $A = R/I$ is Artinian (finite length)

$s := \max\{n : \mathfrak{m}_A^n \neq 0\}$ the socle degree of A.

$t := \dim_K(0 :_A \mathfrak{m}_A)$ is the type of A.

A is Gorenstein if $t = 1$.

The Hilbert function $HF_A : \quad \mathbb{N} \to \mathbb{N}$

$$h_i = HF_A(i) = \mu(\mathfrak{m}^i) = \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

and denote the h-vector

$$h_A = (h_0, h_1, \ldots, h_s)$$

$d = \dim_k A = \sum_{i=0}^{s} h_i$ multiplicity (length)

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Main Goal

Describe the isomorphism classes (or to give information on the structure) of local Artin K-algebras of given length d (or given Hilbert function)

The problem is related to the study of the components of the Hilbert scheme

$$Hilb_d(\mathbb{A}^n)$$

of d points in the affine space $\mathbb{A}^n$. Possible fields of interest:

- Irreducibility of $Hilb_d(\mathbb{A}^n)$
- Rationality of $P_K^A(z)$ (the Poincare' Series)
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Some results

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- B. Poonen, *Contemp. Math.*, vol. 463, (2008)
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A natural question:

The Hilbert function is a first constraint for being isomorphic.

$$A \simeq B \qquad \Longrightarrow \qquad HF_A = HF_B$$

Remark that

$$HF_A = HF_G$$

where

$$G = gr_{\mathfrak{m}}(A) = \oplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$$

is the associated graded ring.

Notice that $gr_{\mathfrak{m}}(A) = K[x_1, \dots, x_n]/I^*$ where I^* is the homogeneous ideal generated by the initial forms of the elements in I.

$$\boxed{I \sim I^* \ ?}$$

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Canonically graded

Following a definition given by J. Emsalem:

Definition

A local algebra $(A, \mathfrak{m})$ is canonically graded if there exists a K-algebra isomorphism between A and its associated graded ring $gr_{\mathfrak{m}}(A)$.

!!!! Consider

$$A = K[[t^2, t^3]] \simeq K[x, y]/(y^2 - x^3)$$

is graded setting $deg(x) = 2$ and $deg(y) = 3$, but it is not canonically graded because A is reduced and $gr_{\mathfrak{m}}(A) = K[x, y]/(y^2)$ is not reduced.

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Example

Consider the Gorenstein local rings A with h-vector:

$$h = (1, 2, 2, 1)$$

We have only two models: $I = \begin{cases} (x^2, y^3) \\ (xy, x^3 - y^3) \end{cases}$

Remark that both the models are homogeneous !!! Hence A is graded.

We will see later (not trivial) that if A is Gorenstein

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Example (Elias, Valla)

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Finitely many isomorphism classes?

- B. Poonen proved: $d = 7$ $\exists\, \infty$ isomorphism classes.
- Elias-Valla proved: $d = 10$ $\exists$ 1-dimensional family of

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$$I_p = (x^3y - px^5, y^2 - x^4)$$

with $p \notin \mathfrak{m}$ and $p^2 - 1 \notin \mathfrak{m}$.

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The dual module: $V = Hom(A, K)$

Let $R = K[[x_1, \dots, x_n]]$ and $P = K[y_1, \dots, y_n]$

$$\left\{ \begin{array}{c} (R/I, \mathfrak{m}) \text{ Artin local rings :} \\ \text{with } \mathrm{socdeg}(R/I) = s. \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{f. g. } R\text{-submodules of } P \\ \text{generated in degree} \leq s \end{array} \right\}$$

Translate the analytic isomorphisms
in terms of the dual module
in an effective framework

In the graded case it is well understood in terms of $GL_n(K)$, see Iarrobino and Kanev's book (Appendix).

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Macaulay's Inverse System

Let

$$R = K[[x_1, \ldots, x_n]] \quad \text{and} \quad P = K[y_1, \ldots, y_n]$$

P has a structure of R-module by the following action

$$\begin{array}{cccc} \circ: & R \times P & \longrightarrow & P \\ & (f, g) & \to & f \circ g = f(\partial_{y_1}, \ldots, \partial_{y_n})(g) \end{array}$$

where ∂_{y_i} denotes the partial derivative with respect to y_i.

Starting from $\circ$ we consider the following pairing of K-vector spaces:

$$\begin{array}{cccc} \langle\ ,\ \rangle: & R \times P & \longrightarrow & K \\ & (f, g) & \to & (f \circ g)(0) \end{array}$$

In the following $(\quad)^*$ means w.r.t. $\langle\ ,\ \rangle$

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Local Macaulay's Inverse System

Let $E = E(A) = (e_1, \dots, \dots, e_s)$ the socle type and $t = \sum_{i=1}^{s} e_i$.

$$\left\{ \begin{array}{c} R/I \text{ is Artin local algebra} \\ \text{with type } t \text{ and socle type } E \end{array} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} M =< f_1, \dots, f_t >_R \\ R\text{-submodule of } P \\ \text{gen. by } e_i \text{ polynomials } f_i \\ \text{of degree } i = 1, \dots, s \end{array} \right\}$$

$$I \longrightarrow I^{\perp} = \{g \in P : I \circ g = 0\} = (R/I)^*$$

$$Ann_R(\underline{F}) = \{g \in R : g \circ f_i = 0\} \longleftarrow M = \langle \underline{F} \rangle_R = \langle f_1, \dots, f_t \rangle_R$$

We will write $A_{\underline{F}} = R/Ann_R(\underline{F})$. Hence if A is Gorenstein

$$A = R/Ann_R(f) := A_f$$

Local Macaulay's Inverse System

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$$\left\{ \begin{array}{c} R/I \text{ is Artin local algebra} \\ \text{with type } t \text{ and socle type } E \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} M =< f_1, \ldots, f_t >_R \\ R\text{-submodule of } P \\ \text{gen. by } e_i \text{ polynomials } f_i \\ \text{of degree } i = 1, \ldots, s \end{array} \right\}$$

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Hilbert function via Inverse system

Graded case: $A = R/I = R/Ann(\underline{F})$

$$HF_{R/I}(i) = \dim_K (I^\perp)_i = \dim_K \langle \partial_{d_j - i} F_j \rangle$$

where ∂_{d_j-i} denotes the partial derivative of order $d_j - i$ with $d_j = degF_j$.

If I is not necessarily homogeneous, we define the following K-vector space:

$$\boxed{\begin{gathered}(I^\perp)_i := \frac{I^\perp \cap P_{\le i} + P_{<i}}{P_{<i}}. \\ HF_{R/I}(i) = \dim_K (I^\perp)_i.\end{gathered}}$$

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$Aut(R/I)$ via Inverse System

Given I and J ideals of R such that $\mathfrak{m}^{s+1} \subset I, J$, let φ be an isomorphism of K-algebras

$$\varphi : R/I = A_{\underline{F}} \xrightarrow{\sim} R/J = A_{\underline{G}}.$$

In particular $\varphi \in Aut_R(R/\mathfrak{m}^{s+1}) \subseteq Aut_K(R/\mathfrak{m}^{s+1})$ and denote by $M(\varphi)$ the associated matrix w.r.t. a basis Ω of $P_{\leq s}$.

$M(\varphi)$ is an element of $Gl_r(K)$ where $r = dim_k(R/\mathfrak{m}^{s+1}) = \binom{n+s}{s}$.

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$$\varphi^* : J^{\perp} = \langle \underline{G} \rangle \longrightarrow I^{\perp} = \langle \underline{F} \rangle$$

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(if $\Omega = \langle x^{\alpha} \rangle$, then $\Omega^* = \langle \frac{1}{\alpha!} y^{\alpha} \rangle$).

φ^* does not act as a substitution !!!

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The classification, up to analytic isomorphism, of the Artin local K-algebras of multiplicity d, socle degree s and embedding dimension n

is equivalent

to the classification, up to the action of $\mathcal{R} \subseteq Aut_K(R/\mathfrak{m}^{s+1})$, of the K-vector subspaces of $P_{\leq s}$ of dimension d, stable by derivations and containing $P_{\leq 1} = K[y_1, \ldots, y_n]_{\leq 1}$.

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Theorem (J. Elias,—)

Let $A = R/I$ be an Artinian Gorenstein local K-algebra with h-vector

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Then A is canonically graded.

"Sketch of the proof" If $A = A_f$ with $f = F_3 + \dots$ lower terms, we prove

$$A_f \simeq A_{F_3}$$

We show that $\forall\ f \in P$ of degree three $\exists \varphi \in Aut(R/\mathfrak{m}^4)$ such that

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Artinian Gorenstein K-algebras of socle degree 3

The Hilbert function of a Gorenstein local algebra of socle degree 3, say A_f with $f = F_3 + \dots$, is not necessarily symmetric:

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If $m > n$ the associated graded ring $gr_{\mathfrak{m}}(A)$ is not longer Gorenstein, but

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The following facts are equivalent:

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Classification

Corollary

The classification of Artinian Gorenstein k-algebras A with h-vector

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Socle degree > 3 ?

We recall that the Gorenstein local rings with Hilbert function

$$(1, 2, 2, 2, 1)$$

are not necessarily graded.

I have anticipated that if the Gorenstein local algebra has Hilbert function

$$(1, 2, 3, 2, 1)$$

then it is graded.

In both cases $gr_{\mathfrak{m}}(A)$ is Gorenstein, but in the second case the local algebra is compressed.

Compressed algebras

Definition

An Artin algebra $A = R/I$ of socle type E is compressed if and only if it has maximal length $e(A) = \dim_K A$ among Artin quotients of R having socle type E and embedding dimension n.

$$A \text{ compressed} \quad \Longrightarrow \quad E(A) = E(gr_{\mathfrak{m}}(A)) \text{ and hence } gr_{\mathfrak{m}}(A) \text{ compressed}$$

It is also known that if A is compressed

$$I^{\perp} = \langle g_1, \ldots, g_t \rangle \quad \Longrightarrow \quad (I^*)^{\perp} = \langle G_1, \ldots, G_t \rangle \text{ the corresp. leading forms}$$

$$(g_i = G_i + ...)$$

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Compressed Gorenstein

Theorem (Elias, —-)

Let A be an Artin compressed Gorenstein local K-algebra.

If $s \leq 4$ then A is canonically graded.

The result cannot be extended to $s = 5$.

Example

Let us consider the ideal

$$I = (x_1^4, x_2^3 - 2x_1^3 x_2) \subset R = K[[x_1, x_2]].$$

The quotient $A = R/I$ is a compressed Gorenstein algebra with

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(3) $s = 4$ and $n = 2$.

The result includes:

- Compressed level algebras of socle degree $s = 3$. (A. De Stefani, *Comm. Algebra*)
- Gorenstein $s = 5$ $\quad h = (1, n, \binom{n+1}{2}, n, 1)$

  More in general: Socle type $(e_1, e_2, e_3, 1)$.

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We have seen that the result cannot be extended to $s = 5$ and $e_4 = 1$.

The result cannot be extended to $s = 4$ and $e_4 > 1$.

Example

Let us consider the forms of degree 4 in $P = K[y_1, y_2, y_3]$:

$$F = y_1^2 y_2 y_3, \quad G = y_1 y_2^2 y_3 + y_2 y_3^3$$

and define in $R = K[[x_1, x_2, x_3]]$ the ideal

$$I = Ann(F + y_3^3, G).$$

Then $A = R/I$ is a compressed level algebra with socle degree 4, type 2 and Hilbert function

$$h = (1, 3, 6, 6, 2).$$

One can prove that $I \not\simeq I^*$.

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