Networks and the Deodhar decomposition of a Grassmannian

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Outline

1) A bit of total positivity history

2) Schubert cell decomposition

$\rightsquigarrow$

Positroid stratification

$\rightsquigarrow$

Deodhar decomposition

3) Go-diagrams and networks

A tiny total positivity example

Def: A matrix is <u>totally positive</u> if all of its minors are positive.

Eg: Consider 2×2 matrices over $\mathbb{R}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Five minors:

$$a, \; b, \; c, \; d, \; \Delta$$

Two minimal positivity tests:

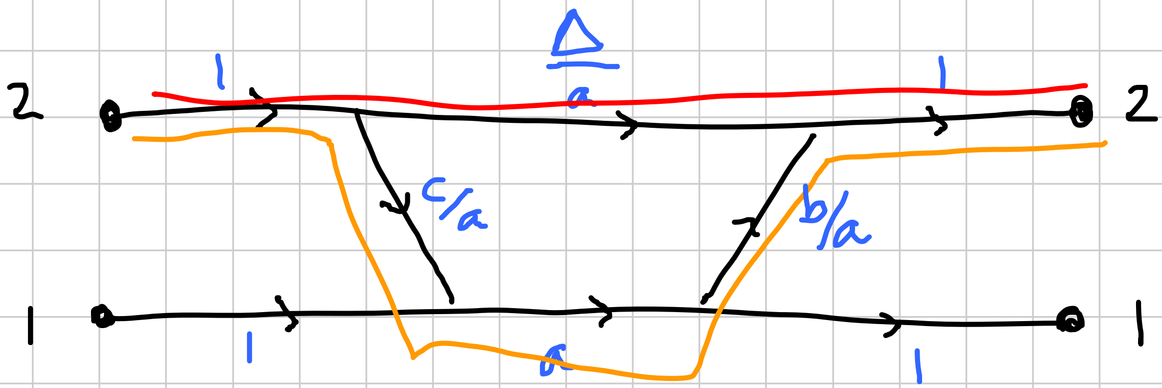
$$\{ a, b, c, \Delta \}$$

$$ad = \Delta + bc$$

$$\{ d, b, c, \Delta \}$$

Using networks to parametrize TP matrices

TP test: $\{\, a,\ b,\ c,\ \Delta \,\}$



$$\begin{array}{cc} & \begin{array}{cc} 1 & \end{array} \\ 1 & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

Weighted path matrix W defined by:

$$w_{ij} = \sum_{\substack{P \text{ is a directed} \\ \text{path from } i \text{ to } j}} wt(P),$$

where $wt(P)$ is the product of the weights of all edges of P.

Networks and TP cont.

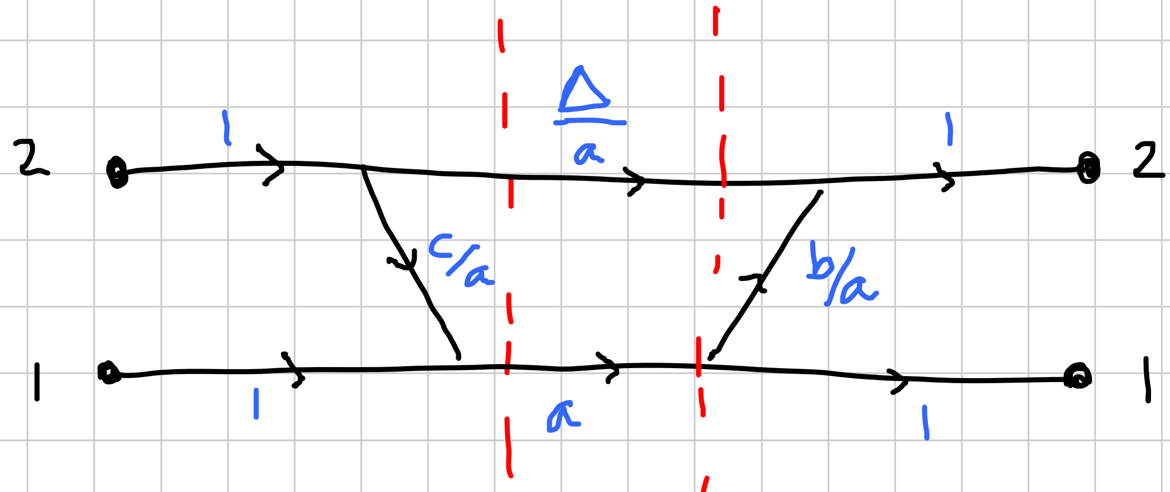
Thm (Lindström, Gessel-Viennot): Minors of W enumerate non-intersecting path families.

Fact: The weights on the network can be chosen to be Laurent monomials in the variables of a single positivity test (i.e. extended cluster).

Consequence: Combinatorial proof of Laurent positivity for $\mathbb{C}[SL_n]$.

Networks and TP, cont.

The network is actually encoding a factorization of our matrix:



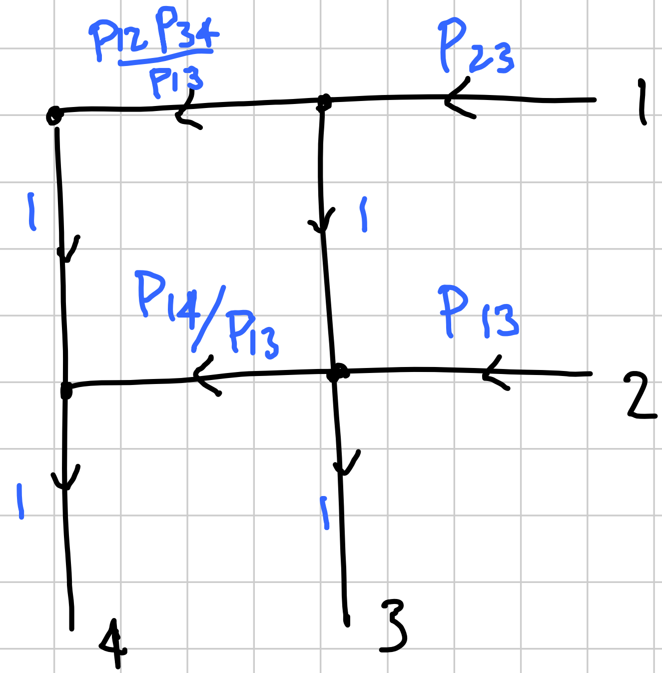
$$\begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{\Delta}{a} \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

See Fomin - Zelevinsky TP: T & P for more.

TP in Grassmannians

We can also use networks to parametrize the TP part of $Gr_{k,n}(\mathbb{R})$, at least in terms of one especially nice cluster.

Eg $Gr_{2,4}(\mathbb{R})$



Some decompositions of $Gr_{k,n}(\mathbb{R})$

Schubert cell decomposition $\rightsquigarrow$ Positroid stratification $\rightsquigarrow$ Deodhar decomposition

cells indexed by partitions

strata indexed by "Ⅎ-diagrams"

components indexed by "Go-diagrams"

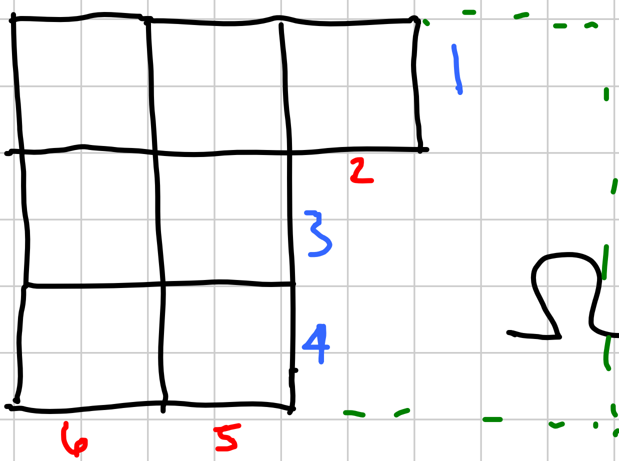
All can be characterized by saying that certain Plücker coordinates vanish, certain others do not. (Many are not specified.)

Schubert cell decomposition of $Gr_{k,n}(\mathbb{R})$:

Points in $Gr_{k,n}(\mathbb{R})$ represented by full rank $k \times n$ matrices. In row echelon form, we can "see" a partition.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$
$$\begin{bmatrix} 1 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

this tells us our Schubert cell.

$\lambda =$ 

$I(\lambda) = \{1, 3, 4\}$.

Ω_λ = subset of $Gr_{k,n}$ where $P_{I(\lambda)}$ is the lex min nonzero Plücker coord.

Positroid stratification of $Gr_{k,n}(\mathbb{R})$

Take the common refinement of the n "cyclically shifted Schubert cell decompositions".

Usual one has $1 < 2 < \ldots < n$.

Shifted ones do the same thing with respect to the order

$$a <_a a+1 <_a a+2 <_a \cdots <_a n <_a 1 <_a \cdots <_a a-1.$$

Characterized by
$$\begin{cases} P_{I_1} & \text{lex min WRT } <_1 \\ P_{I_2} & \text{" " " } <_2 \\ \vdots & \vdots \\ P_{I_n} & \text{" " " } <_n \end{cases}$$

$\{I_1, \ldots, I_n\}$ = Grassmann necklace

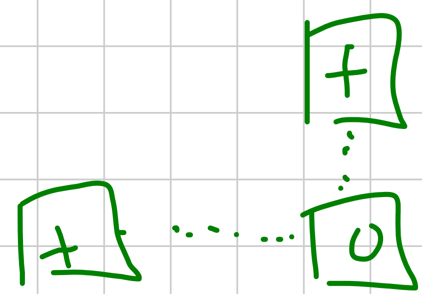
Grassmann necklace

$\Big\downarrow$

⅃-diagram

filling of Young diagram w/ + and 0 such that

We avoid



Deodhar decomposition of $Gr_{k,n}(\mathbb{R})$

Outline:

- Start with the Marsh-Rietsch parametrizations for Deodhar components of the flag variety, and project to the Grassmannian.
- Encode this with a "Go-diagram" (from Kodama-Williams work on KP-solitons)
- Theorem 1: Using the Go-diagram, we can draw a network parametrizing the Deodhar component.
- Theorem 2: Deodhar components can be characterized by vanishing/non-vanishing of certain Plücker coordinates.

The Marsh-Rietsch parametrization for $Gr_{k,n}(\mathbb{R})$

Assume we are in Ω_λ.

11	10	9	8
7	6	5	
4	3	2	
1			

s_4	s_3	s_2	s_1
s_5	s_4	s_3	
s_6	s_5	s_4	
s_7			

$s_i = (i, i+1)$

$$w = s_7 \, s_4 \, s_5 \, s_6 \, s_3 \, s_4 \, s_5 \, s_1 \, s_2 \, s_3 \, s_4$$

(reduced)

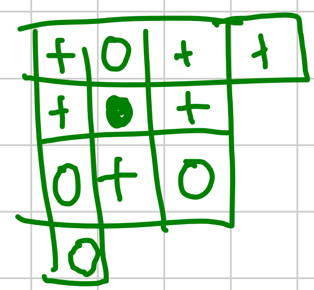
M-R param, cont

Look for distinguished subwords of w.

Rule: Working L→R, if choosing s_i would decrease the length of the subword so far, you must take it.

$W = \underline{s_7}\ \underline{s_4}\ s_5\ \underline{s_6}\ s_3\ \underline{s_4}\ s_5\ s_1\ s_2\ \underline{s_3}\ s_4$

forced



To make a Go-diagram,

record + if you don't choose the letter

o if you do and length so far increases

• " " " " " " decreases

MR-param, cont.

MR characterization: All points in the Deodhar component corresponding to D fit the same factorization pattern.

$$\boxed{+} \rightsquigarrow y_i(p), \quad p \in \mathbb{R}^*$$

$$\boxed{0} \rightsquigarrow \dot{s}_i$$

$$\boxed{\bullet} \rightsquigarrow x_i(m)\, \dot{s}_i^{-1} \qquad m \in \mathbb{R}$$

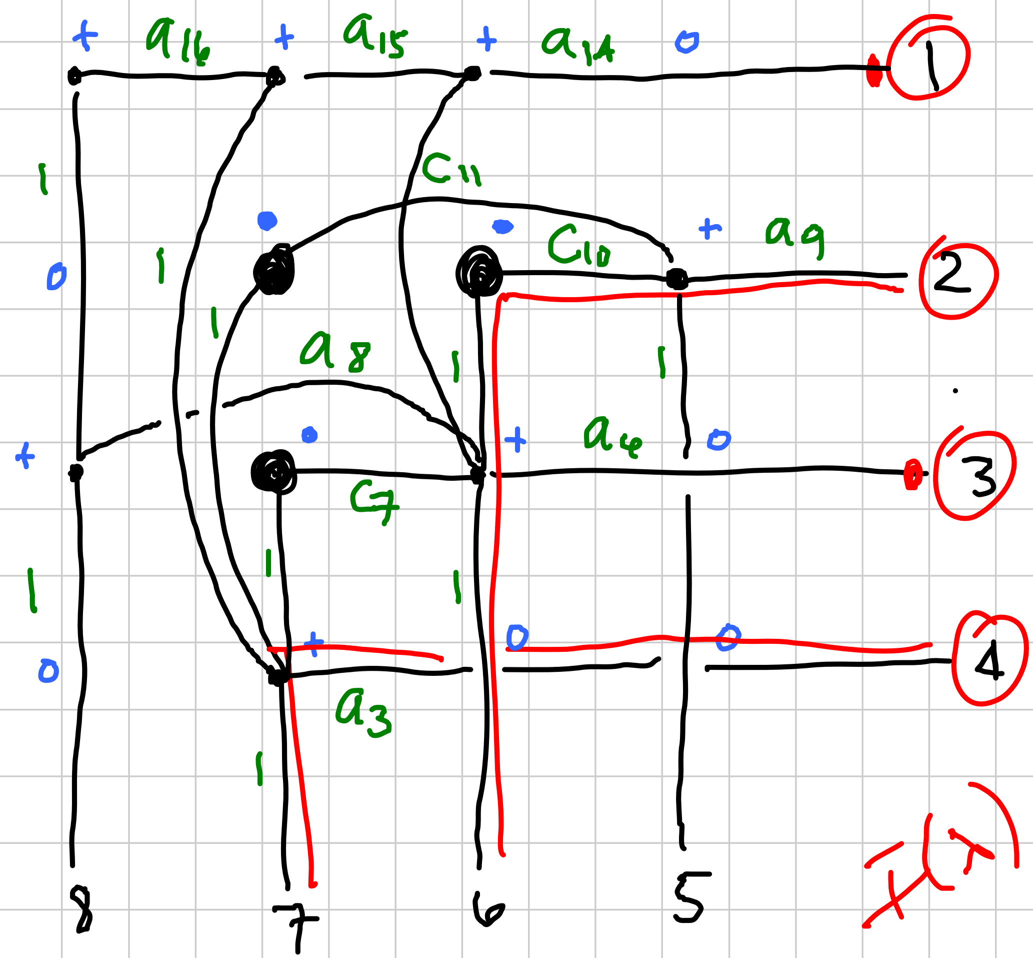
$$\begin{pmatrix} 1 & & & \\ & 1 & \circledast & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\begin{array}{lccccccccccc} W = & \underline{s_7} & \underline{s_4} & s_5 & \underline{s_6} & s_3 & \underline{s_4} & s_5 & s_1 & s_2 & \underline{s_3} & s_4 \\ & 0 & 0 & + & 0 & + & \bullet & + & + & + & 0 & + \end{array}$$

$$\dot{s}_7\, \dot{s}_4\, y_5(p_3)\, \dot{s}_6\, y_3(p_5)\, x_4(m_6)\, s_4^{-1} \cdots$$

A network corresponding to a Go-diagram

Eg: $D = \begin{array}{|c|c|c|c|} \hline + & + & + & 0 \\ \hline 0 & \bullet & \bullet & + \\ \hline + & \bullet & + & 0 \\ \hline 0 & + & 0 & 0 \\ \hline \end{array}$

$N_D =$ 

$Gr_{4,8}(\mathbb{R})$

$J = \{1, 3, 6, 7\}$

$a_i \in \mathbb{R}^*$, $c_i \in \mathbb{R}$

directed

Theorem 1 (Talaska-Williams)

The network N_D parametrizes the Deodhar component S_D corresponding to D, in the sense that we get a bijection

$$\left\{ \begin{array}{l} \text{choices of weights} \\ a_i \in \mathbb{R}^* \\ c_i \in \mathbb{R} \quad \text{on } N_D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{points in} \\ S_D \end{array} \right\},$$

where $P_J = \sum \pm \mathrm{wt}(\mathcal{F})$

$\mathcal{F}$ = NI path fam from $I(\lambda)$ to J.

<u>Corollary</u> Every point in $Gr_{k,n}(\mathbb{R})$ can be represented by a unique network coming from a Go-diagram.

Theorem 2 (Talaska-Williams)

The Deodhar component S_D, where D is of shape λ, is characterized by:

$$A \in S_D \text{ iff } \begin{cases} P_{I(\lambda)}(A) \neq 0 & \\ P_J(A) = 0 & \text{if } J <_{lex} I(\lambda) \\ P_{I_b}(A) \neq 0 & \text{if } b = \boxed{+} \\ P_{I_b}(A) = 0 & \text{if } b = \boxed{0}, \end{cases}$$

No cond on $\boxed{\bullet}$

where each box b in D has a special minor I_b associated to it (a bit technical to describe).