IMPLICITIZATION OF RATIONAL SURFACES: EASY ALGORITHMS, DEEP PROOFS

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THE PROBLEM WE WANT TO SOLVE

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- *Let f be a rational parametrization of a (hyper)***surface** $\mathcal{S} = (F = 0) \subset \mathbb{A}^3$, $F \in \mathbb{K}[T_1, T_2, T_3]$.

$$\begin{array}{ccc} \mathbb{A}^2 & \overset{f}{\dashrightarrow} & \mathbb{A}^3 \\ s = (s_1, s_2) & \mapsto & \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}, \frac{f_3(s)}{f_0(s)}\right) \end{array}$$

- $f_i \in \mathbb{K}[s_1, s_2]$ *with* $\gcd(f_0, \ldots, f_3) = 1$ *and F is irreducible with* $F(f(s)) = 0$ *(whenever defined).*
- **We assume the parametrization** f **known but the implicit equation** F **not known**
- *That, is, we want to switch from parametric to implicit representations of rational surfaces.*

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PREDICTING THE MONOMIALS THAT COULD OCCUR IN F

- The convex hull of the exponents of the monomials ocurring in a non zero polynomial h is called the Newton polytope $\mathrm{N}(h)$ of h.
- When h is a polynomial in (s_1, s_2) of degree (at most) d, its Newton polytope $\mathrm{N}(F)$ is (contained in) the triangle with vertices $(0,0), (d,0), (0,d)$. Its Euclidean area is $\frac{d^2}{2}$ and its lattice area is $2\,\frac{d^2}{2} = d^2$.

THEOREM (CLASSIC, [BUSÉ-JOUANOLOU'03])

For generic polynomials $f_0, \ldots, f_3$ of degree d, the degree of F is d^2 and its Newton polytope is the triangle with vertices $(0,0,0), (d^2,0,0), (0,d^2,0), (0,0,d^2)$.

THEOREM ([STURMFELS-YU'94])

For generic polynomials $f_0, \ldots, f_3$ with fixed Newton polytope P, the degree of F is the lattice area v of P and its Newton polytope is the triangle with vertices $(0,0,0), (v,0,0), (0,v,0), (0,0,v)$.

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FIRST NAIVE ALGORITHM

- Assume the Newton polytope $\mathrm{N}(F)$ of F is known (as in the previous theorems) and number $m_1, \ldots, m_N \in \mathbb{N}^3$ the integer (lattice) points in $\mathrm{N}(F)$.
- Consider indeterminates $c = (c_1, \ldots, c_N)$ and write $F = \sum_{i=1}^{N} c_i T^{m_i}$.
- Substitute $T = f(s)$ and equate to 0 the coefficient of each power of (s_1, s_2) that occurs.
- This sets a system $\mathcal{L}$ of linear equations in c, with solution space of dimension 1. **Any nonzero solution** c **will give a choice of implicit equation** F.

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WHICH IS THE SIZE OF THE LINEAR SYSTEM

LEMMA

In case f_i are generic polynomials of degree d in (s_1, s_2), the number of unknowns in $\mathcal{L}$ is $\binom{d^2+3}{3}$ ($\approx d^6/6$) and the number of equations is $\binom{d^3+2}{2}$ ($\approx d^6/2$).

LEMMA

For any lattice polygon P and generic polynomials f_i with Newton polytope P, the linear system $\mathcal{L}$ has $\binom{\mathrm{vol}_{\mathbb{Z}}(P)+3}{3}$ ($\approx \frac{\mathrm{vol}_{\mathbb{Z}}(P)^3}{6}$) variables and $\frac{\mathrm{vol}_{\mathbb{Z}}(P)^3}{2} + \frac{\mathrm{vol}_{\mathbb{Z}}(P)^2}{2} \mathrm{vol}_{\mathbb{Z}}(\partial P) + 1$ equations ($\approx \frac{\mathrm{vol}_{\mathbb{Z}}(P)^3}{2}$).

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IMPLICITIZE $T - f(s) = 0$ MEANS ELIMINATING THE s-VARIABLES

GENERAL STRATEGIES TO ELIMINATE VARIABLES

- Reduce the problem to a linear algebra problem
- Hide the variables one wants to eliminate in the (monomial) basis.
- Use determinants.

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Restating the problem: find a representation matrix

A rational surface $\mathcal{S}$ is given as the closed image of a map

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where the f_i are polynomials such that $\gcd(f_0, \ldots, f_3) = 1$.

DEFINITION

A matrix representation M of $\mathcal{S}$ is a matrix with entries in $\mathbb{K}[T_1, T_2, T_3]$, generically of full rank, such that the rank of $M(P)$ drops iff the point P lies on $\mathcal{S}$.
Moreover, the greatest common divisor of all minors of M of maximal size equals $F^{\deg(f)}$.

Having the matrix M is sufficiently good for many purposes (like checking if a point lies on the surface), well adapted for numerical computations and cheaper to get!

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MAIN TOOLS

LINEAR SYZYGIES

Given polynomials $f_0, \ldots, f_3 \in \mathbb{K}[s_1, s_2]$, a syzygy on $f_0, \ldots, f_3$ is a 4-tuple of polynomials $(h_0, \ldots, h_3)$ such that $\sum_{i=0}^{3} h_i f_i = 0$.

MONOMIAL STRUCTURE OF THE INPUT POLYNOMIALS

Study supports of local cohomology over toric rings (embedded or not), extending results of Busé, Chardin and Jouanolou for the homogeneous case.

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- The use of syzygies for the implicitization of (conic) surfaces goes back to Steiner in 1832.
- Meyer in 1887 described syzygies of 3 polynomials and made a general conjecture proved by Hilbert in 1890.
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Suite

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MOTIVATION: MATRIX REPRESENTATIONS FOR CURVES

- A planar rational curve $\mathcal{C}$ over a field $\mathbb{K}$ is given as the image of a map

$$\begin{array}{ccc} \mathbb{P}^1 & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \\ s & \mapsto & (f_0(s) : f_1(s) : f_2(s)), \end{array}$$

$f_i \in \mathbb{K}[s]$ homogeneous polynomials of degree d in s, $gcd(f_0, f_1, f_2) = 1$.

- A (linear) syzygy can be represented as a linear form $L = h_0 T_0 + h_1 T_1 + h_2 T_2$ in the new variables $T = (T_0, T_1, T_2)$ with $h_i \in \mathbb{K}[s]$ such that

$$\sum_{i=0,1,2} h_i f_i = 0.$$

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- A (linear) **syzygy** can be represented as a linear form $L = h_0 T_0 + h_1 T_1 + h_2 T_2$ in the new variables $T = (T_0, T_1, T_2)$ with $h_i \in \mathbb{K}[s]$ such that

$$\sum_{i=0,1,2} h_i f_i = 0.$$

MOTIVATION: MATRIX REPRESENTATIONS FOR CURVES

- $\mathrm{Syz}(\phi)$ = { all linear syzygies}. For $\nu \in \mathbb{N}$, the graded part $\mathrm{Syz}(\phi)_\nu$ $(\deg(h_i) \leq \nu)$ is a $\mathbb{K}$-vector space with dimension $N(\nu) < \infty$.
- **Attention: here comes the main elimination step:**
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- Let M_ν be the $N(\nu) \times (\nu + 1)$ matrix of coefficients of the L_i's with respect to a $\mathbb{K}$-basis of $\mathbb{K}[s]_\nu$:

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- If $\nu = d - 1$, then M_ν is a square matrix, such that $\det(M_\nu) = F^{\deg(f)}$, where F is an implicit equation of $\mathcal{C}$.
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Linear syzygies, quadratic syzygies, . . . , implicit equation

A common shape

- Linear syzygies of degree ν: $H(s,T) = \sum_{i=0}^{3} h_i(s)T_i$ such that $\sum_{i=0}^{3} h_i(s)f_i(s) = 0$. Thus, $\deg(H)$ in s variables is ν, $\deg(H)$ in T variables is 1.
- Quadratic syzygies of degree ν': $H(s,T) = \sum_{i\leq j=0}^{3} h_{i,j}(s)T_iT_j$ such that $\sum_{i,j=0}^{3} h_{i,j}(s)f_if_j(s) = 0$. Thus, $\deg(H)$ in s variables is ν', $\deg(H)$ in T variables is 2.
- Implicit equation (of degree D): $H(s,T) = \sum_{|\alpha|\leq D} h_\alpha T^\alpha$ such that $\sum_\alpha h_\alpha f^\alpha(s) = 0$. Thus, $\deg(H)$ in s variables is 0, $\deg(H)$ in T variables is D.
- So to go from linear syzygies to the implicit equation we play the game of lowering the degree in the s variables to 0 (which increases the degree in the T varibles up to (at most) D)!

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THE METHOD DISTILLED IN SIMPLEST TERMS

BASIC GENERAL ALGORITHM [BDD] SPARSE, [BCJ] CLASSIC

- **INPUT:** $(f_0(s), f_1(s), f_2(s), f_3(s))$ with Newton polytopes contained in P (a lattice polygon in the first quadrant), satisfying suitable hypotheses.
- **STEP 1:** Consider syzygies $(h_0, \ldots, h_3)$ with $N(h_i) \subset 2P = \{p_1 + p_2, p_i \in P\}$. Let $(h_0^{(j)}, , \ldots, h_3^{(j)}), j = 1, \ldots, N$, be a $\mathbb{K}$-basis of such syzygies.
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Theorem([Botbol-D.-Dohm])

If the "suitable hypotheses" are satisfied, the matrix M is a representation matrix for the closed image $\mathcal{S}$ of f: its rank drops precisely at the points of $\mathcal{S}$ and the gcd of its maximal minors equals $F^{\deg(f)}$.

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SIZE OF THE MATRICES IN THE ALGORITHM

- Assume for example that P is the triangle of size d. Then, in fact, it is enough to consider syzygies of degree $2d-2$. Therefore, to find them, we have a system on $4\binom{2d}{2}$ variables with $\binom{3d}{2}$ equations. That is, both sizes, as well as the vector space dimension of the space of syzygies in this degree, are quadratic in d.
- The matrix M has then a number of rows quadratic in d. The number of its columns equals $\binom{2d}{2}$, again quadratic in d.

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IMPROVEMENTS [BDD],[BOTBOL],[B-CHARDIN]

- The general algorithm above can be refined (under the same hypotheses of f).
- If the lattice polytope P can be written as $P = dP'$, with P' another lattice polygon without interior lattice points, then we can consider in STEP 1 syzygies $(h_0, \dots, h_3)$ with $N(h_i)$ contained in $(2d-1)P'$, which it is strictly contained in $2P$, that is, with smaller support.
- Moreover, in case P' is the unit simplex, it is enough to consider syzygies with support inside $(2d-2)P'$.
- In general, the presence of base points $(V(f_0, \dots, f_3) \neq \emptyset)$ allows to take syzygies with smaller supports.
- In the bihomogeneous case (that is, when P is a rectangle), a detailed study of regularity allows to get the following improvement in the support of the proposed syzygies in STEP 1.

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THEOREM

The output of the following algorithm is a representation matrix for the rational surface parametrized by a rational map which in the conditions stated in the INPUT below.

- **INPUT:** (f_0, f_1, f_2, f_3) *with Newton polytopes contained in a rectangle* R*, say, with opposite vertices* $(0,0)$ *and* (a,b)*, satisfying certain hypotheses.*
- **STEP 1:** *Consider syzygies* $(h_0, \ldots, h_3)$ *with* $N(h_i) \subset R'$ *the rectangle with opposite vertices* $(0,0)$ *and* $(2a, b)$ *(or* $(0,0)$ *and* $(a, 2b)$*). Let* $(h_0^{(j)}, , \ldots, h_3^{(j)}), j = 1, \ldots, N$*, be a* $\mathbb{K}$*-basis of such syzygies.*
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- We can instead consider the map $\tilde{f} : \mathbb{A}^2 \dashrightarrow \mathbb{P}^3$ with image inside 3-dimensional projective space given by

$$s \mapsto (f_0(s) : f_1(s) : f_2(s) : f_3(s)).$$

  The defining equation of the closure $\widetilde{S}$ of the image of $\tilde{f}$ is the homogenization of the polynomial F with a new variable T_0.

- We can consider the rational parametrization from another normal algebraic variety $\mathcal{T}$ which contains the domain of $\tilde{f}$ as a dense subset (a toric variety), so we get $\tilde{f} : \mathcal{T} \dashrightarrow \mathbb{P}^3$.

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- The equation of the closed image $\widetilde{\mathcal{S}}$ of $\tilde{f} : \mathcal{T} \dashrightarrow \mathbb{P}^3$ on the variables $(T_0, \ldots, T_3)$ depends on the relation between the polynomials $f_0, \ldots, f_3 \in A$, a ring with another set of variables. The natural ambient for our elimination problem is a variety where both group of variables are involved:

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  where Γ is the closure of the graph of $\tilde{f}$. Thus, $\widetilde{\mathcal{S}} = \pi_2(\Gamma)$.

- $\Gamma \subset \mathcal{T} \times \mathbb{P}^3$ corresponds to $A[T_0, T_1, T_2, T_3] \twoheadrightarrow \mathrm{Rees}_A(I)$, the Rees algebra of I over A. The projection $\pi_2(\Gamma)$ corresponds to eliminating the variables in A.

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FROM $\mathrm{Rees}_A(I)$ TO $\mathrm{Sym}_A(I)$

- There is no "universal" way to compute a free presentation for $\mathrm{Rees}_A(I)$. Thus, in general one approximates $\mathrm{Rees}_A(I)$ by the symmetric algebra $\mathrm{Sym}_A(I)$, that admits a known resolution (under some hypotheses).
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- But which is the relation between $\mathrm{Rees}_A(I)$ and $\mathrm{Sym}_A(I)$?
- Assume for every $p \in V$, I_p is a complete intersection in A_p (I is lci, generated locally by 2 elements). Then, $\mathrm{Rees}_A(I)$ and $\mathrm{Sym}_A(I)$ define the same scheme in $\mathcal{T} \times \mathbb{P}^3$.
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- The **approximation complex** $\mathcal{Z}_\bullet$ is a bi-graded complex of $A[\underline{T}]$-modules constructed by means of two Koszul complexes (with respect to $\tilde{f}$ and to T), which gives a resolution of $\mathrm{Sym}_A(I)$ if I is a lci (or even alci).
- For any given degree ν in the source variables, it induces a graded complex $(\mathcal{Z}_\bullet)_\nu$ of $\mathbb{K}[\underline{T}]$-modules

  $0 \to (\mathcal{Z}_3)_\nu \xrightarrow{\bar{e}_3} (\mathcal{Z}_2)_\nu \xrightarrow{\bar{e}_2} (\mathcal{Z}_1)_\nu \xrightarrow{\bar{e}_1} (\mathcal{Z}_0)_\nu$ and M_ν is the matrix of $\bar{e}_1$ in the monomial bases.
- What does this imply for us? If the approximation complex gives a presentation of $\mathrm{Sym}_A(I)$, M_ν represents $\mathcal{S}$ for ν beyond the torsion of $\mathrm{Sym}_A(I)$ (because for these degrees the Rees and Symmetric algebra coincide).
- To bound this torsion: either study the embedded toric variety associated to the input lattice polygon P (cut out by the corresponding toric ideal) (the associated ring A is Cohen Macaulay since P is always normal in dimension two), or study multigraded regularity in the Cox ring associated to the normal fan of P ([Maclagan-Smith]).

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TESTING THE HYPOTHESES?

We'd need to check that there are finitely many isolated base points of $\tilde{f}$ which are local complete intersection (I is lci of the right dimension) …

- Dimension can be checked, lci in particular cases.
- But what if we don't check this and run the algorithm? …nothing bad!
- We just check whether M_ν has full rank (by evaluation).
  - If the rank is not maximal, then there is a base point p which is not an almost local complete intersection. (I_p cannot be generated by 3 elements).
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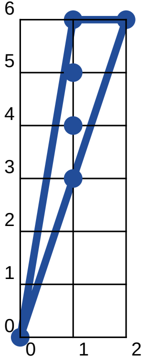
- We just check whether M_ν has full rank (by evaluation).
  - If the rank is not maximal, then there is a base point p which is not an almost local complete intersection. (I_p cannot be generated by 3 elements).
  - If the rank is maximal, it may happen that the rank of M_ν drops at some other places besides $\mathcal{S}$ due to the existence of an almost local complete intersection base point which is not a complete intersection. In this case, the approximation complex is still exact but there might be extra factors in the gcd from these points.

THE METHOD INTO ACTION

- Consider the parametrization with 6 monomials: $(f_0, f_1, f_2, f_3) = (2 + s^2t^6, st^6 + 2, st^5 - 3st^3, st^4 + 5s^2t^6)$
- $\mathrm{N}(f) = P =$



THE METHOD INTO ACTION

- $(f_0,f_1,f_2,f_3) = (2+s^2t^6, st^6+2, st^5-3st^3, st^4+5s^2t^6)$
- Coordinate ring of $\mathcal{T}$ is $\mathbb{K}[X_0,\ldots,X_5]/J_P$, where $J_P = (X_3^2-X_2X_4, X_2X_3-X_1X_4, X_2^2-X_1X_3, X_1^2-X_0X_5)$
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- The matrix M_2 is a matrix representation of size 17×34.
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AND THE IMPLICIT EQUATION IS ...

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The reatest common divisor of the 17-minors of the 17×34 matrix M_2 is the homogeneous implicit equation of the surface:

$$\begin{aligned}
&2809T_0^2T_1^4 + 124002T_1^6 - 5618T_0^3T_1^2T_2 + 66816T_0T_1^4T_2 + 2809T_0^4T_2^2 \\
&-50580T_0^2T_1^2T_2^2 + 86976T_1^4T_2^2 + 212T_0^3T_2^3 - 14210T_0T_1^2T_2^3 + 3078T_0^2T_2^4 \\
&+13632T_1^2T_2^4 + 116T_0T_2^5 + 841T_2^6 + 14045T_0^3T_1^2T_3 - 169849T_0T_1^4T_3 \\
&-14045T_0^4T_2T_3 + 261327T_0^2T_1^2T_2T_3 - 468288T_1^4T_2T_3 - 7208T_0^3T_2^2T_3 \\
&+157155T_0T_1^2T_2^3T_3 - 31098T_0^2T_2^3T_3 - 129215T_1^2T_2^3T_3 - 4528T_0T_2^4T_3 \\
&-12673T_2^5T_3 - 16695T_0^2T_1^2T_3^2 + 169600T_1^4T_3^2 + 30740T_0^3T_2T_3^2 \\
&-433384T_0T_1^2T_2T_3^2 + 82434T_0^2T_2^2T_3^2 + 269745T_1^2T_2^2T_3^2 + 36696T_0T_2^3T_3^2 \\
&+63946T_2^4T_3^2 + 2775T_0T_1^2T_3^3 - 19470T_0^2T_2T_3^4 + 177675T_1^2T_2T_3^3 \\
&-85360T_0T_2^2T_3^3 - 109490T_2^3T_3^3 - 125T_1^2T_3^4 + 2900T_0T_2T_3^4 \\
&+7325T_2^2T_3^4 - 125T_2T_3^5
\end{aligned}$$

Or set $T_0 = 1$ to get the affine equation.

IN PRESENCE OF (GOOD) BASE POINTS

- Consider the following example:

$$(s_1, s_2) \mapsto \left(\frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_0}\right),$$

$$f_0 = 1 - s_1 s_2; \quad f_1 = -s_1^{36} s_2 + 1; \quad f_2 = -s_2(-s_1^{38} + s_2); \quad f_3 = s_1^{37} - s_2.$$

- In this case, the syzygy method we studied is fast.
- Gröbner basis elimination methods do not terminate.
- Resultant methods also fail, because there is a base point in the torus.

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Thank you for your attention!