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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu **Email/Phone:** aseceleanu2@math.unl.edu

Speaker's Name: Lauren Williams

Talk Title: Introduction to cluster algebras I

Date: 08/22/12 **Time:** 9:00 am / pm (circle one)

List 6-12 key words for the talk: cluster algebra, positivity, grassmanian

Please summarize the lecture in 5 or fewer sentances: The lecture explains what a cluster algebra is and provides relevant examples.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Lauren Williams   Cluster algebras – I

- Introduced by Fomin & Zelevinsky ~2000
- motivated by work of Lusztig on dual canonical bases and total positivity

Cluster algebra portal → to find links, refs, conferences

Facts:
- A cluster algebra is a certain subalgebra of $k(x_1, \dots, x_n)$ (the field of rational functions)
- Generators are constructed inductively by a process called mutation

Def An $n \times n$ integer matrix $B = (b_{ij})$ is skew-symmetrizable if $\exists\, d_1, \dots, d_n \in \mathbb{Z}^+$ s.t. $d_i b_{ij} = -d_j b_{ji}$ $\forall i,j$
(in particular a skew-symm. matrix is skew-symmetriz.)

Can associate a cluster algebra $\mathcal{A}(B)$ to B
Start w/ seed $\Sigma = (\{x_1, \dots, x_n\}, B)$

cluster variable   cluster   exchange matrix

From Σ, can apply an involution called mutation in each of n directions, obtaining n more seeds.
Columns of B encode exchange relations:
- for $k \in \{1, \dots, n\}$, $x_k x_k' = \prod_{b_{ik} > 0} x_i^{|b_{ik}|} + \prod_{b_{ik} < 0} x_i^{|b_{ik}|}$

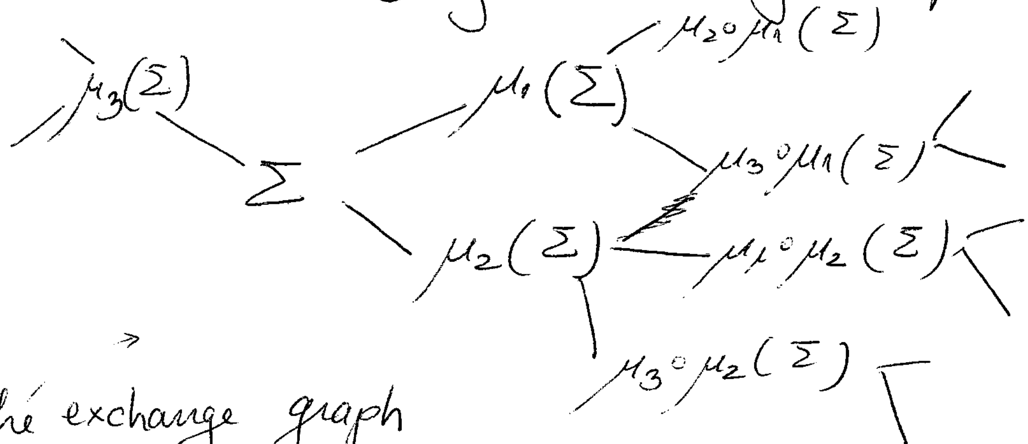
(this defines a new cluster variable x_k')
- for $k \in \{1, \dots, n\}$, $\exists$ another seed
$\mu_k(\Sigma) = (\{x_1, x_2, \dots, \hat{x}_k, \dots, x_n\} \cup \{x_k'\}, \mu_k(B))$ where

$$\mu_k(B) = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} & \text{if } b_{ik} b_{kj} \le 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

Rk: $\mu_k(B)$ is skew-symmetrizable
$\mu_k \circ \mu_k = \mathrm{Id}$

Start from the initial seed Σ and apply all possible sequences of mutations. This produces the set of all cluster variables (may be an infinite process).

$n=3$



the exchange graph

<u>Def</u> A cluster algebra $\mathcal{A}(B)$ is a subalgebra of $k(x_1, \dots x_n)$ generated by all cluster variables

<u>Ex1</u>: Rank 2 cluster algebras

Let $\mathcal{F} = k(x_1, x_2)$

Fix $b, c \in \mathbb{Z}^+$.

Define $B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$. (all skew-symmetrizable matrices are of this form)

Then $x_1 x_1' = x_2^c + 1$. Refer to x_1' as x_3. (i.e. $x_3 = x_1' = \dfrac{x_2^c + 1}{x_1}$)

$x_2 x_2' = x_1^b + 1$. Refer to x_2' as x_0.

$$\mu_1(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}, \quad \mu_2(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

$$\mu_1^2 = \mu_2^2 = id$$

$$\mu_1 \circ \mu_2(B) = \mu_2 \circ \mu_1(B) = B$$

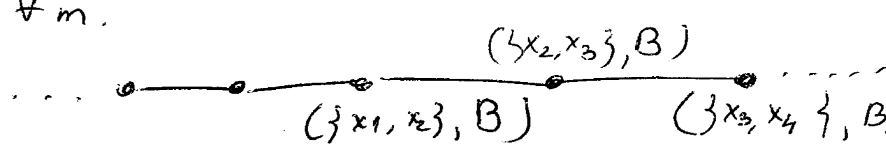
Our cluster variables are $\{x_m\}_{m \in \mathbb{Z}}$

They satisfy the exchange relations:

$$x_{m-1} x_{m+1} = \begin{cases} x_m^b + 1, & m = \text{odd} \\ x_m^c + 1, & m = \text{even} \end{cases}$$

The cluster algebra $\mathcal{A}(b, c)$ is a subring of $\mathcal{F}$ generated by the x_m, $\forall m$.

Exchange graph:



Note: every cluster var. can be expressed as a rational function in the initial cluster.

Laurent phenomenon (Fomin - Zelevinsky):
Given any seed $(\{x_1, \dots, x_n\}, B)$ for any cluster alg & any cluster variable x, one can express x as a Laurent poly in the original variables.

Positivity conjecture (F-Z)
All coefficients in these Laurent polynomials are positive.

Ex 1a: Let $b = c = 1$. Start with $\{x_1, x_2\}$.

Then $x_3 = \dfrac{x_2+1}{x_1}$, $x_4 = \dfrac{x_3+1}{x_2} = \dfrac{\frac{x_2+1}{x_1}+1}{x_2} = \dfrac{x_1+x_2+1}{x_1 x_2}$

$$x_5 = \frac{x_4+1}{x_3} = \frac{\frac{x_1+x_2+1}{x_1x_2}+1}{\frac{x_2+1}{x_1}} = \frac{x_1+x_2+1+x_1x_2}{x_2(x_2+1)} = \frac{x_1+1}{x_2}$$

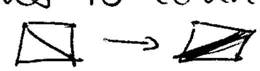
$$x_6 = \frac{x_5+1}{x_4} = \frac{\frac{x_1+1}{x_2}}{\frac{x_1+x_2+1}{x_1x_2}} = \frac{(x_1+x_2+1)\cdot x_1}{x_1+x_2+1} = x_1$$

$(x_7 = x_2)$

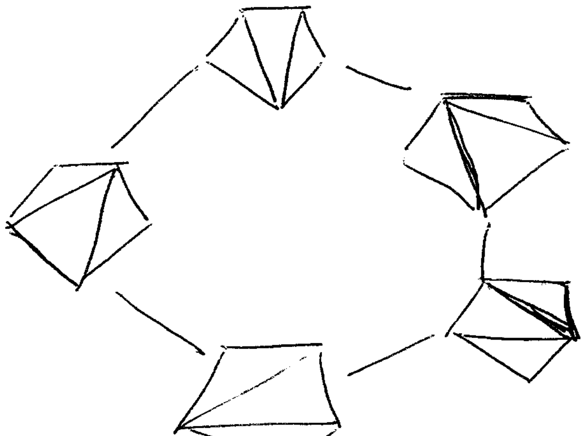
This is a cluster alg. w/ finite number of variables. i.e. a finite type cluster algebra.

Thm (F-Z)
Finite type cluster algebras can be classified using Dynkin diagrams.

Ex 2: Consider a polygon with $n+3$ sides and choose any triangulation T. (it has $n+3$ boundary segments and n diagonals).

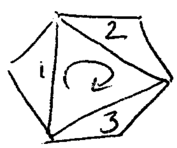
Note: the set of all triangulations is connected by elementary moves called flips: 

Eg (5 triangulations of the pentagon)



Can associate an $n \times n$ matrix $B(T)$ to T:
First label diagonals of T by $1, \dots, n$:

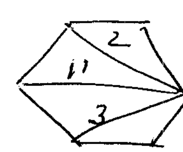
$B(T) = (b_{ij})$ where $b_{ij} =$ # triangles w sides i and j with j following i in clockwise order
$-$ # triangles w/ sides i and j with j following i in counter-clockwise order

Eg $T =$ 

$\rightsquigarrow B(T) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & -1 & 0 \end{array}$

This gives a cluster alg. $\mathcal{A}(B(T))$ assoc. to each triangulation of a polygon.

$$\mu_1(B(T)) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Also, let's flip the diagonal "1" in T: 

$= T'$

Claim: Let T be a triangulation and T' be a new triangulation obtained by flipping the diagonal i. Then $B(T') = \mu_i(B(T))$

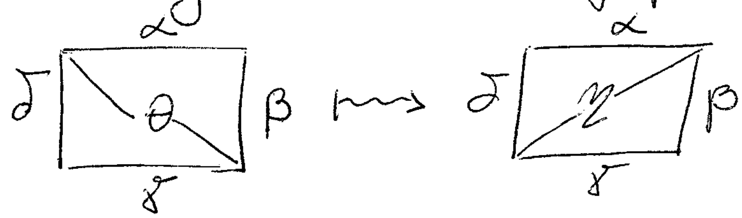
Corollary: The cluster algebra $\mathcal{A}(B(T))$ does not depend on T, only on the number $n+3$.

<u>Thm</u> We have bijections:

cluster variables x_γ $\longleftrightarrow$ diagonals γ of the $(n+3)$-gon

clusters $\longleftrightarrow$ triangulations

exchange relations $\longleftrightarrow$ flips



$$x_\theta x_\eta = x_\alpha x_\gamma + x_\beta x_\delta$$

More on mutation:

if B is skew-symmetric, can think of B as the adjacency matrix of a directed graph.

<u>Ex</u> $B = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \longleftrightarrow Q = \begin{tikzcd} \overset{1}{\circ} \arrow[r, shift left] \arrow[r, shift right] & \overset{2}{\circ} \\ & \underset{3}{\bullet} \arrow[u] \end{tikzcd}$

mutation μ_k of B $\longleftrightarrow$ operation on Q:

- reverse all arrows incident to vertex k
- for each directed path of length 2 in Q going $i \to k \to j$ add new arrow $i \to j$

(also: two arrows in opposite directions cancel each other out)

<u>Ex</u> $\mu_2\left(\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet} \right) = \begin{tikzcd} \underset{1}{\bullet} \arrow[rr, bend left] & \underset{2}{\bullet} \arrow[l] & \underset{3}{\bullet} \arrow[l] \end{tikzcd}$