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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Karin Baur

Talk Title: Cluster algebras, quiver mutation and triangulations

Date: 08/22/12 Time: 2:00 am / pm (circle one)

List 6-12 key words for the talk: cluster algebra, triangulation, quiver.

Please summarize the lecture in 5 or fewer sentances: The talk explains the manner in which cluster algebras arise from triangulations of surfaces

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Karin Baur – Cluster algebras, quiver mutations and triangulations

Poisson geom — total positivity / canonical bases

cluster algebras — cluster categories

...

triangulations of surfaces, polygons

Fomin – Shapiro – Thurston 2006

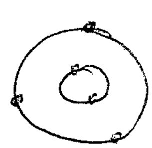
I Surfaces
II Quivers of a triangulation
III Quiver $\rightsquigarrow$ cluster alg.

$\Downarrow$ cluster category

IV Arcs in surfaces $\rightsquigarrow$ (cluster) categories

I <u>Surfaces</u> (S, M)

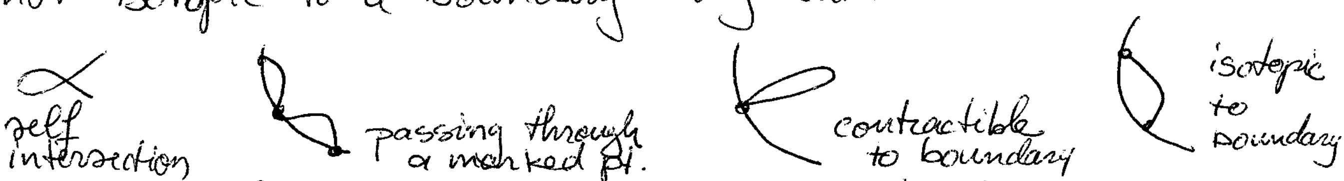
S = connected 2-dim Riemann surface w/ boundary
M = marked pts : on boundary, in interior (punctures)
Exclude : spheres w/ < 3 punctures
monogon w/ < 2 punctures 
digon 
triangle

<u>Main examples</u> :
- n-gon ($M = \{p_1, \ldots, p_n\}$) ($n \geq 4$)
- punctured n-gon ($n \geq 2$) 
- annulus  n_1, n_2 points on boundaries

(S, M) (up to homeomorphism) is determined by
- the genus g

- the # of boundary components
- ~~the~~ # of marked pts on these components
- the # of punctures g

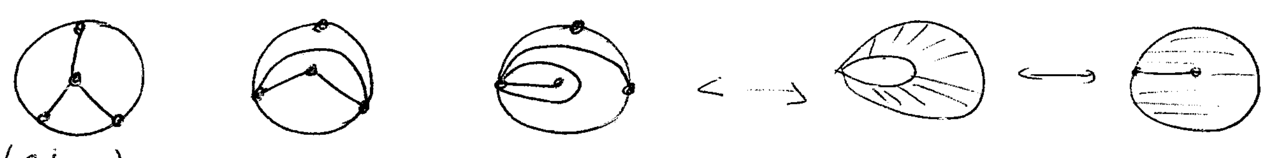
An <u>arc</u> in (S,M) (up to isotopy) is a curve in S whose endpoints belong to M and has no self-intersections ~~and~~, does not pass through a marked ~~point~~ other than the end points and does not contract to the boundary and is not isotopic to a boundary segment.



There are finitely many such arcs $\Leftarrow$ (S,M) is $\begin{cases} n\text{-gon } (n \geq 4) \\ \text{punctured } (n \geq 2) \\ n\text{-gon} \end{cases}$

Fock - Goncharov
"Handbook of Teichmüller theory"

<u>Def</u> A triangulation of (S,M) is a max. collection of distinct arcs that pairwise don't cross.
This cuts (S,M) into triangles, whose vertices and sides might coincide.



1 star

3 triangulations of this type

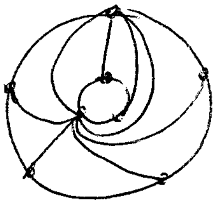
6 triang.

<u>Def</u> Rank of (S,M) = # arcs needed in a triangulation (depends only on the surface)

$$rk(S,M) = 6g + 3b + 3g + \underset{\uparrow}{n} - 6$$
$$n_1 + \dots + n_b$$

<u>surface</u>	<u>rank</u>
n-gon	$n-3$
punctured n-gon	n
annulus	$n_1 + n_2$

__Eg__ (annulus)



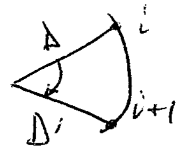
1 __Rk__ Any two triangulations of (S,M) can be related by a sequence of __flips__.
(Harer - Hatcher)

Flip : D = diagonal in triangulation T
$D \overset{\text{flip}}{\longleftrightarrow} D'$

$\longrightarrow T \setminus D \cup D'$

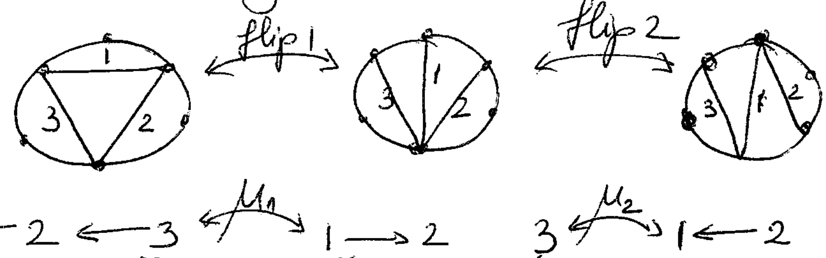
II Quiver of T (of (S,M))

$Q(T)$: • vertices = arcs in T
• arrows :



• cancel 2-cycles

__Example__ :



$$\begin{tikzcd} 1 \arrow[rr, bend right] & 2 \arrow[l] & 3 \arrow[l] \arrow[r, leftrightarrow, "\mu_1"] & 1 \arrow[r] & 2 & 3 \arrow[ll, bend left] \arrow[r, leftrightarrow, "\mu_2"] & 1 & 2 \arrow[l] & 3 \arrow[ll, bend left] \end{tikzcd}$$

__Mutation of quivers__

Q : no loops
no 2-cycles } assumption for the rest of talk

$k \in Q$ vertex
$\mu_k(Q)$ = mutation of Q at k is given by :

a) for every subquiver $i \to k \to j$ add arrow $i \to j$
b) reverse all arrows incident w k
c) 2-cycles are cancelled

III Quiver $\rightsquigarrow$ cluster algebras (A)
$\rightsquigarrow$ cluster category (B)

$\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$
quiver w/ n vertices (no loops, no 2-cycles)
A <u>seed</u> (Q, u), $u = \{u_1, \dots, u_n\}$, $u_i \in \mathcal{F}$ generate $\mathcal{F}$
mutation of seed (Q, u) at $k = (Q', u')$ where $Q' = \mu_k(Q)$
u': $u_i' = u_i$ for $i \neq k$

$$u_k' \cdot u_k = \prod_{i \to k} u_i + \prod_{k \to i} u_i$$

(Q', u') is again a seed
$\mu_k(\mu_k(Q, u)) = (Q, u)$

<u>Example</u>: $\left(1 \to 2, \{x_1, x_2\}\right) \xrightarrow{\mu_2} \left(1 \leftarrow 2, \left\{x_1, \frac{x_1+1}{x_2}\right\}\right)$

$\xrightarrow{\mu_1} \left(1 \to 2, \left\{\frac{x_2+1}{x_1}, x_2\right\}\right)$

(five seeds)

....

(A) $(Q, \{x_1, \dots, x_n\})$ initial seed
- a <u>cluster</u> for this $\{u_1, \dots, u_n\}$ appearing in a seed (arbitrary sequence of mutations)
- <u>cluster variable</u> an element of a seed
- <u>cluster algebra</u> $\mathcal{A}_Q$ = span of all cluster variables (subalg. of $\mathcal{F}$).

<u>Rk</u>: finitely many quivers (under mut.) $\overset{\text{Buan Reiten}}{\Longleftrightarrow}$ Q is Dynkin or extended Dynkin

(finitely many cluster variables $\overset{FZ}{\Longleftrightarrow}$ Dynkin type)

(B) Q as before
kQ = path
$\mathcal{D}^b(kQ\text{-mod})$ = bounded derived category
$[1]$ shift functor
τ AR translate $\Big\}$ $F := \tau^{-1} \circ [1]$

BMRRT $\mathcal{C}_Q := \mathcal{D}^b(kQ\text{-mod}) / F$

Caldero - Chapoton - Schiffler (Type A_{n-3})

Qdiag $\begin{cases} \text{diagonals in } n\text{-gon} \\ \text{rotations } (i,j) \to (i,j+1) \\ \qquad\qquad\quad (i,j) \to (i+1,j) \\ \tau : (i,j) \to (i-1,j-1) \end{cases}$ $\rightsquigarrow \mathcal{C}_Q$

indecomposable objects $\longleftrightarrow$ diag
irred morphisms $\longleftrightarrow$ rotations.