MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Craig Huneke

Talk Title: Introduction to uniformity in commutative algebra #1

Date: 08/27/12 Time: 10:30 am / pm (circle one)

List 6-12 key words for the talk: Hilbert Syzygy Theorem, projective dimension, regularity, resolutions

Please summarize the lecture in 5 or fewer sentances: This lecture discusses resolutions, invariants of resolutions such as projective dimension and regularity and some conjectures about them.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Craig Huneke – Introduction to Uniformity in Commutative Algebra

"In the last twenty years commutative algebra has undergone an intensive development" (Zariski – Samuel)

Thm k field, $S = k[x_1, \dots, x_n] = \bigoplus_{i \geq 0} S_i$. Every ideal in S is finitely generated.

Question ①: is there an absolute bound to the number of generators of ideals in S?

Ans: $n = 1 \Rightarrow S = k[x_1]$ is a PID, hence 1 is a bound for the number of generators of ideals of S

$n \geq 2$ no bound exists

e.g $(x_1, x_2)^N$ has $N+1$ generators

①' What if we take into account the degrees of generators?

② What about up to nilradical $\sqrt{I} = \{ f \mid f^s \in I \}$?

③ What about prime ideals?

④ Are there any effective lower bounds?

③ $n = 2 \Rightarrow 2$ is a lower bound for the number of generators of prime ideals in $k[x_1, x_2]$.

$n \geq 3 \Rightarrow$ there is no such bound (Macaulay)

② Yes; every ideal is generated up to radical by n elements (Storch; Eisenbud – Evans '70)

④ Def codim $I := \dim S - \dim S/I = n - \dim S/I$

Thm (Krull)

codim I is a lower bound for the number of generators of I (it is in fact a sharp bound).

① Assume I is homogeneous.

If I is generated in degrees $\leq d$, then I has at

most the number of generators of $m^d = (x_1, \ldots, x_n)^d$, i.e
$\dim S_d = \binom{n+d-1}{d-1}$.

Def: The function $h_S(d) := \dim S_d$ is called the Hilbert function.
The series $H_S(z) = \sum (\dim S_d) z^d = \frac{1}{(1-z)^n}$ is called the

Hilbert series.

Let M be a finitely generated graded S-module.
$M = \bigoplus_{i \in \mathbb{Z}} M_i$, $S_i M_j \subseteq M_{i+j}$

$h_M(i) := \dim_k M_i < \infty$
$H_M(z) = \sum (\dim_k M_i) z^i$

Thm (Hilbert)
There exists a polynomial $P_M(t) \in \mathbb{Q}[t]$ s.t.
$P_M(i) = h_M(i)$ $\forall i \gg 0$.

Given a short exact sequence $0 \to N \to M \to K \to 0$
we want the morphisms to have degree 0, i.e.
if $f: M \to N$, we want $f(M_i) \subseteq N_i$,
Then $0 \to N_i \to M_i \to K_i \to 0$ is again exact.
If $\deg f = d$, by $S(-d)$ we denote S with generators
in degree S i.e $S(-d)_i = S_{i-d}$
Then $0 \to S(-d) \xrightarrow{f} S \to S/fS \to 0$ is exact, degree 0.

$$\Rightarrow h_{S/fS}(i) = h_S(i) - h_{S(-d)}(i) = h_S(i) - h_S(i-d)$$
$$= \binom{i+n-1}{n-1} - \binom{i-d+n-1}{n-1} \quad \text{for } i \geq d$$

Thm (Hilbert Syzygy Theorem)
$\exists$ a finite free graded resolution of any finitely generated graded module M:
Let $0 \to F_n \to \cdots \to F_0 \to M \to 0$ be a minimal resolution.
$F_i = S(-e_1) \oplus \cdots \oplus S(-e_s)$ or better $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ where

β_{ij} are the <u>graded Betti numbers</u> and they are well-defined since $\beta_{ij} = \dim_k \mathrm{Tor}_i^S(M,k)_j$.

homological degree   internal degree

The resolution of k is the <u>Koszul complex</u> on $x_1, \dots, x_n$;

$$0 \to S(-n)^{\binom{n}{n}} \to \dots \to S(-2)^{\binom{n}{2}} \to S(-1)^n \to S \to k \to 0$$

$$\Rightarrow \beta_{ij}(k) = \begin{cases} 0 & j \neq i \\ \binom{n}{i} & j = i \end{cases}$$

Taking the alternating sum of dimensions in the minimal free resolution:

$$h_M(i) = \sum_{\ell=0}^{n} (-1)^\ell \sum_j \binom{n+i-j}{n-1} \beta_{\ell j}$$; polynomial for large enough i

$$H_M(z) = \frac{\sum_j \left(\sum_\ell (-1)^\ell \beta_{\ell j}\right) z^j}{(1-z)^n}$$

Betti table:

	0	1	2	......
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	.....
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	.....
⋮	⋮	⋮	⋮	

e.g. for the Koszul complex

	0	1	2	....	n
0	1	n	$\binom{n}{2}$		1

for S/fS

	0	1
0	1	
⋮		
$d-1$		1

4 invariants that can be read off a Betti table:

1. $\dim M := \dim(S/\mathrm{ann}\, M) = \deg P_M + 1$
2. $e(M) := (\dim M - 1)! \cdot$ (leading coefficient of P_M)
   ↳ multiplicity
3. $\mathrm{pd}(M) :=$ length of the Betti table (index of the last non-zero column)
4. $\mathrm{reg}(M) :=$ width of the Betti table (index of the last non-zero row)

What type of uniformity might exist for these invariants?

$\operatorname{codim} I \leq \operatorname{pd} S/I \leq n$

Bounds in terms of degrees of generators of I:

<u>Stillman's Question</u>:

Fix the degrees and number of generators of I.

Is $\operatorname{pd}(S/I)$ bounded?

Same question for $\operatorname{reg}(S/I)$.

— These two questions are equivalent (Caviglia)