**Joint Introductory Workshop**

MSRI, Berkeley, August 2012

# Introduction to cluster algebras

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## Main references

Cluster algebras I–IV:

*J. Amer. Math. Soc.* **15** (2002), with A. Zelevinsky;
*Invent. Math.* **154** (2003), with A. Zelevinsky;
*Duke Math. J.* **126** (2005), with A. Berenstein & A. Zelevinsky;
*Compos. Math.* **143** (2007), with A. Zelevinsky.

$Y$-systems and generalized associahedra,
*Ann. of Math.* **158** (2003), with A. Zelevinsky.

Total positivity and cluster algebras,
*Proc. ICM*, vol. 2, Hyderabad, 2010.

## Cluster Algebras Portal

⟨http://www.math.lsa.umich.edu/~fomin/cluster.html⟩

Links to:

- $>$400 papers on the arXiv;
- a separate listing for lecture notes and surveys;
- conferences, seminars, courses, thematic programs, etc.

## Plan

1. Basic notions

2. Basic structural results

3. Periodicity and Grassmannians

4. Cluster algebras in full generality

Tutorial session (G. Musiker)

# FAQ

- Who is your target audience?
- Can I avoid the calculations?
- Why don't you just use a blackboard and chalk?
- Where can I get the slides for your lectures?
- Why do I keep seeing different definitions for the same terms?

# PART 1: BASIC NOTIONS

## Motivations and applications

Cluster algebras: a class of commutative rings equipped with a particular kind of combinatorial structure.

Motivation: algebraic/combinatorial study of *total positivity* and *dual canonical bases* in semisimple algebraic groups (G. Lusztig).

Some contexts where cluster-algebraic structures arise:

- Lie theory and quantum groups;
- quiver representations;
- Poisson geometry and Teichmüller theory;
- discrete integrable systems.

## Total positivity

A real matrix is *totally positive* (resp., *totally nonnegative*) if all its minors are positive (resp., nonnegative).

Total positivity is a remarkably widespread phenomenon:

- 1930s-40s (F. Gantmacher–M. Krein, I. Schoenberg)
  *classical mechanics, approximation theory*

- 1950s-60s (S. Karlin, A. Edrei–E. Thoma)
  *stochastic processes, asymptotic representation theory*

- 1980s-90s (I. Gessel–X. Viennot, Y. Colin de Verdière)
  *enumerative combinatorics, graph theory*

- 1990s-2000s (G. Lusztig, S. F.-A. Zelevinsky)
  *Lie theory, quantum groups, cluster algebras*

## Totally positive varieties
(informal concept)

| | |
|---|---|
| $X$ | complex algebraic variety |
| $\mathbf{\Delta}$ | collection of "important" regular functions on $X$ |
| $X_{>0}$ | *totally positive variety* (all functions in $\mathbf{\Delta}$ are $> 0$) |
| $X_{\geq 0}$ | *totally nonnegative variety* (all functions in $\mathbf{\Delta}$ are $\geq 0$) |

Example: $X = \mathrm{GL}_n(\mathbb{C})$, $\mathbf{\Delta} = \{\text{all minors}\}$.

Example: the *totally positive/nonnegative Grassmannian*.

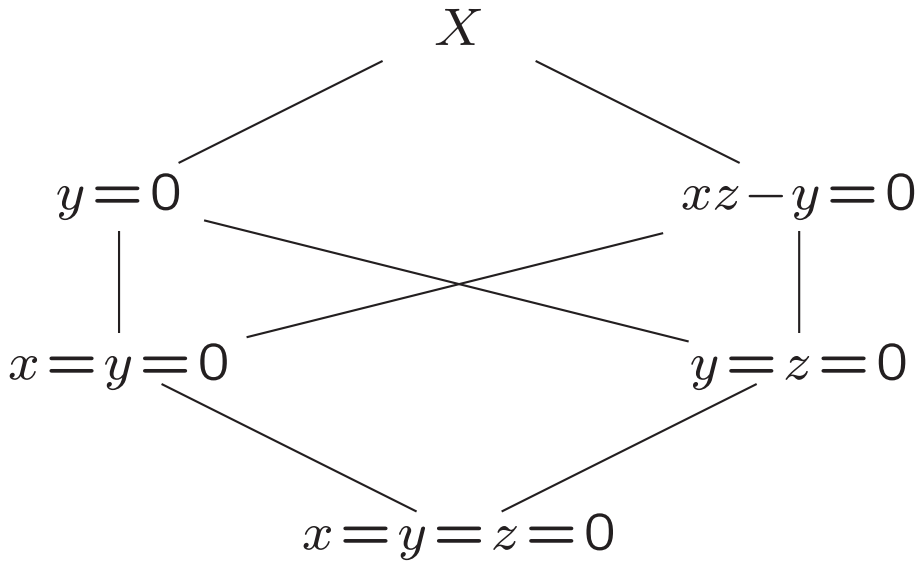
## Why study totally positive/nonnegative varieties?

**(1)** The structure of $X_{\geq 0}$ as a semialgebraic set can reveal important features of the complex variety $X$.

Example: unipotent upper-triangular matrices.

$$X = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$X_{\geq 0} = \left\{ \begin{array}{c} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ xz - y \geq 0 \end{array} \right\}$$



## Why study totally positive/nonnegative varieties? (continued)

**(2)** Some of them can be identified with important spaces.

Examples: decorated Teichmüller spaces (R. Penner, S.F.-D.T.); "higher Teichmüller theory" (V. Fock–A. Goncharov); Schubert positivity via Peterson's map (K. Rietsch, T. Lam).

**(3)** Potential interplay between the *tropicalization* of a complex algebraic variety $X$ and its positive part $X_{>0}$.

## From positivity to cluster algebras

Which algebraic varieties $X$ have a "natural" notion of positivity?

Which families $\mathbf{\Delta}$ of regular functions should be used for that?

The concept of a cluster algebra can be viewed as an attempt to answer these questions.

## Prototypical example of a cluster algebra

Consider the algebra

$$\mathcal{A} = \mathbb{C}[\mathsf{SL}_n]^N \subset \mathbb{C}[x_{11}, \ldots, x_{nn}]/\langle \det(x_{ij}) - 1 \rangle$$

of polynomials in the matrix entries of an $n \times n$ matrix $(x_{ij}) \in \mathsf{SL}_n$ which are invariant under the natural action of the subgroup

$$N = \left\{ \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \subset \mathsf{SL}_n(\mathbb{C})$$

(multiplication on the right).

$\mathcal{A}$ is the *base affine space* for $G = \mathsf{SL}_n(\mathbb{C})$.

## Flag minors and positivity

Invariant theory: $\mathcal{A} = \mathbb{C}[\mathsf{SL}_n]^N$ is generated by the *flag minors*

$$\Delta_I : x \mapsto \det(x_{ij} | i \in I, j \leq |I|),$$

for $I \subsetneq \{1, \ldots, n\}$, $I \neq \emptyset$.

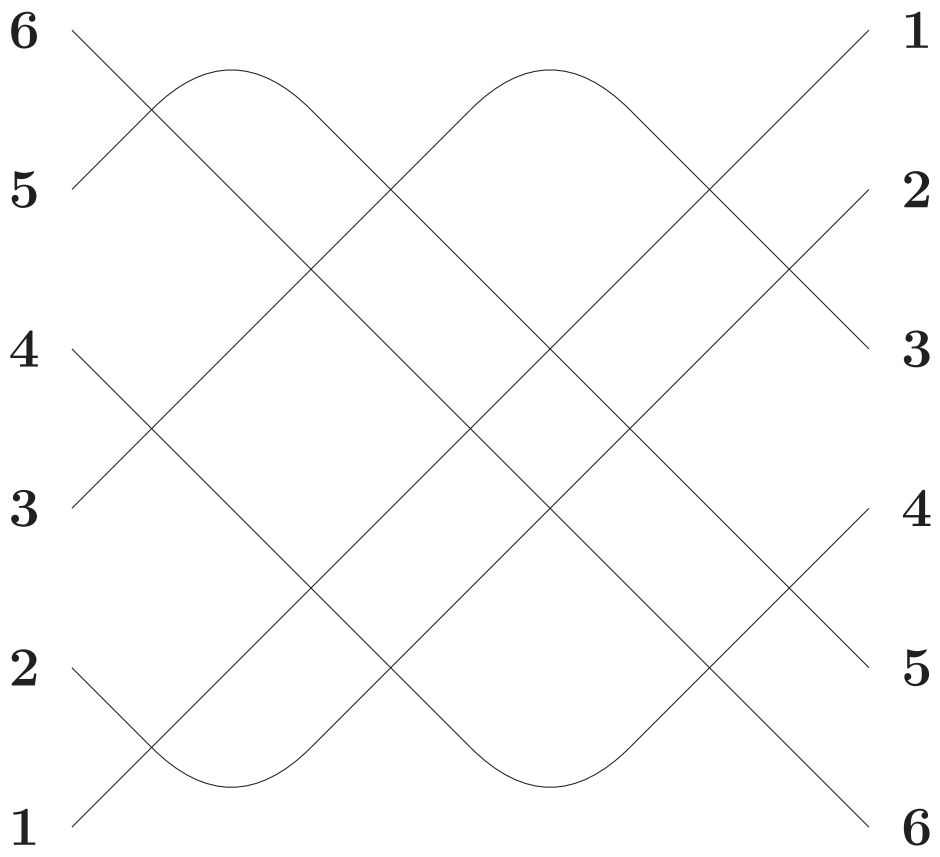
The flag minors $\Delta_I$ satisfy *generalized Plücker relations*.

A point in $G/N$ represented by a matrix $x \in G$ is *totally positive* (resp., *totally nonnegative*) if all flag minors $\Delta_I$ take positive (resp., nonnegative) values at $x$.

There are $2^n - 2$ flag minors. How many of them should be tested to determine whether a given point is totally positive?

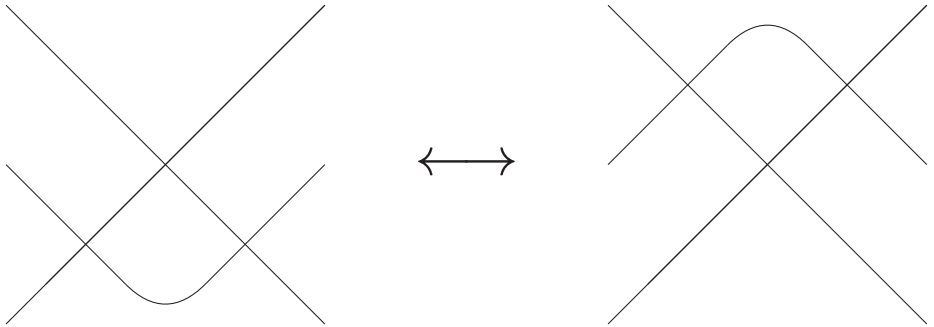
Answer: enough to check $\dim(G/N) = \frac{(n-1)(n+2)}{2}$ flag minors.

# Pseudoline arrangements

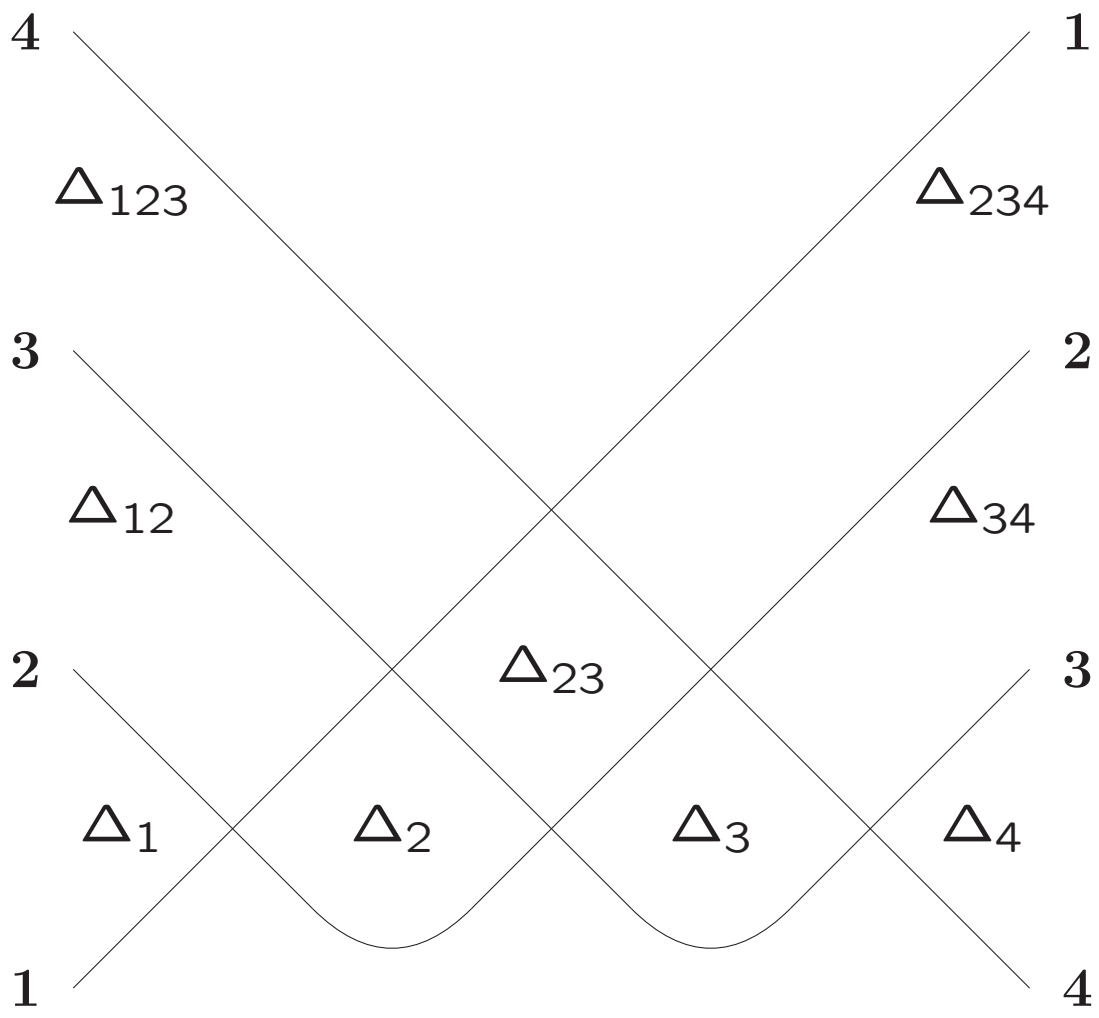


## Braid moves

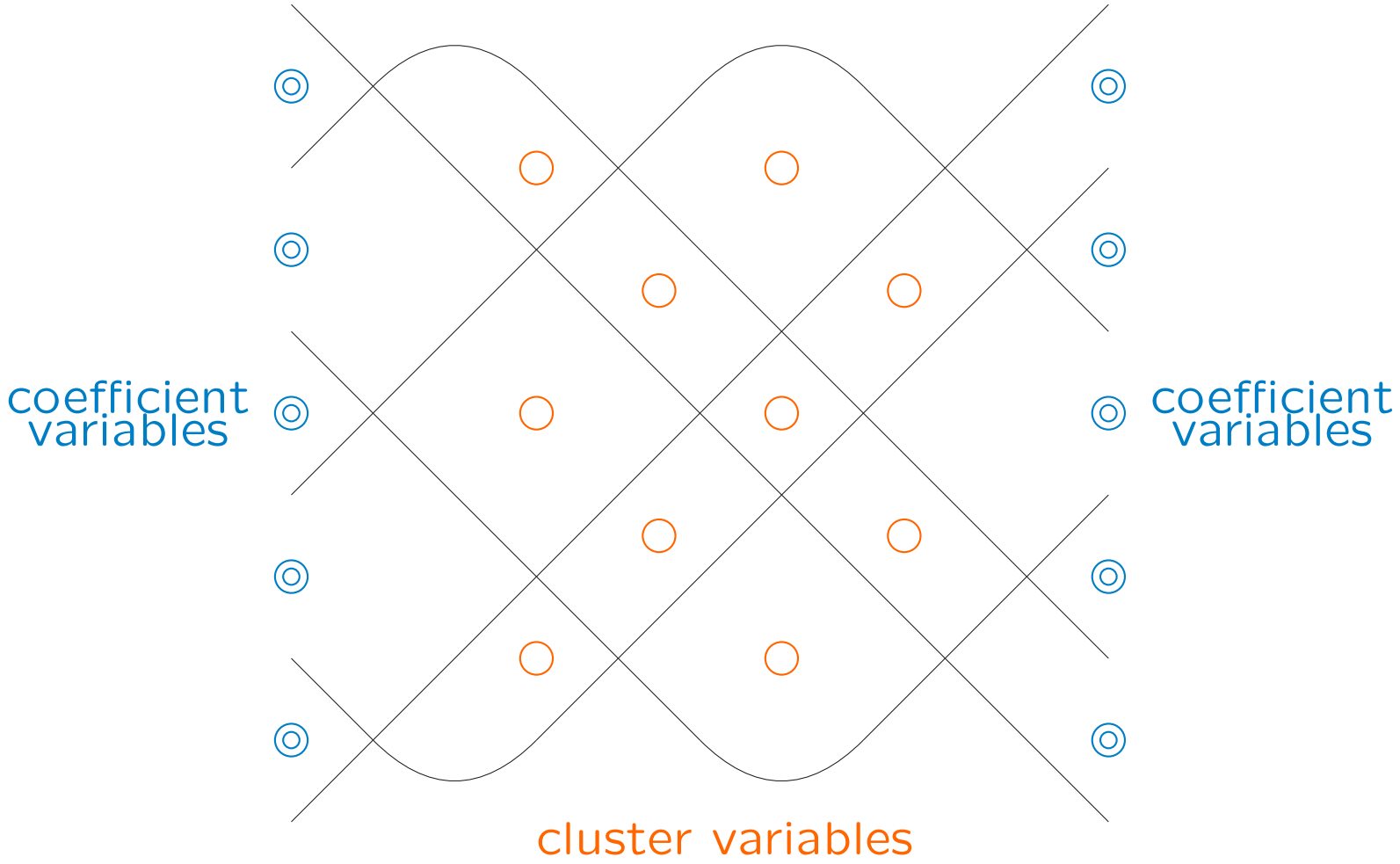
Any two pseudoline arrangements are related by *braid moves:*



# Chamber minors



# Cluster jargon

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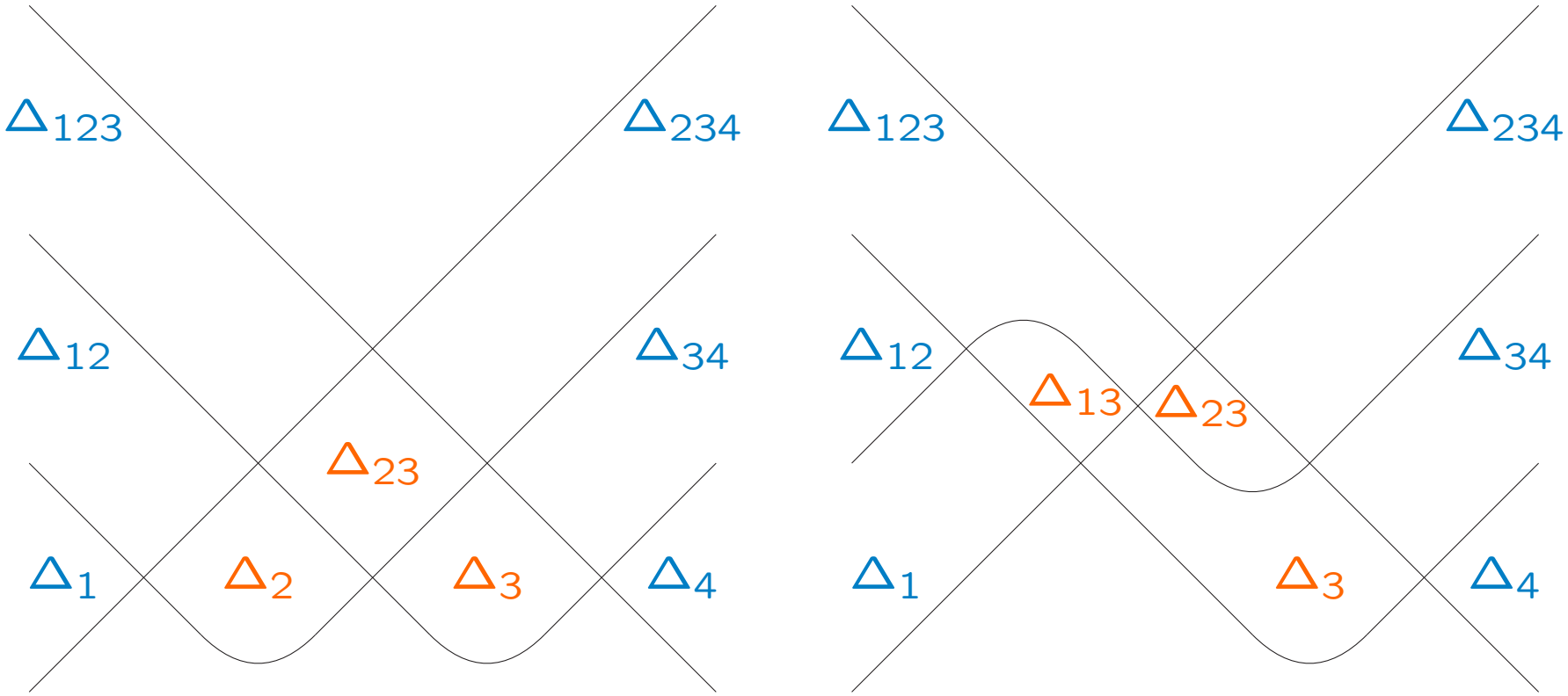
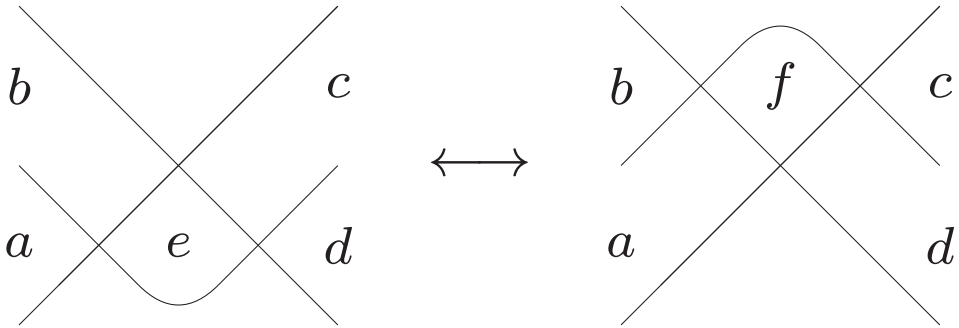
cluster = {cluster variables}

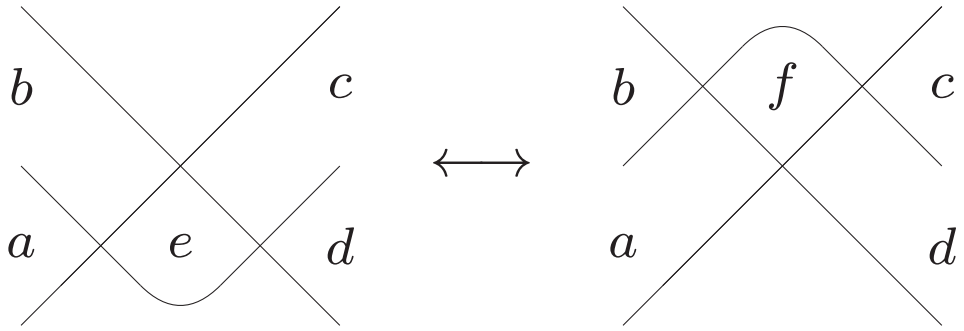
extended cluster = {coefficient variables, cluster variables}

# Braid moves and cluster exchanges



## Exchange relations




The chamber minors $a, b, c, d, e, f$ satisfy the *exchange relation*

$$ef = ac + bd.$$

(For example, $\Delta_2\Delta_{13} = \Delta_{12}\,\Delta_3 + \Delta_1\,\Delta_{23}$.)

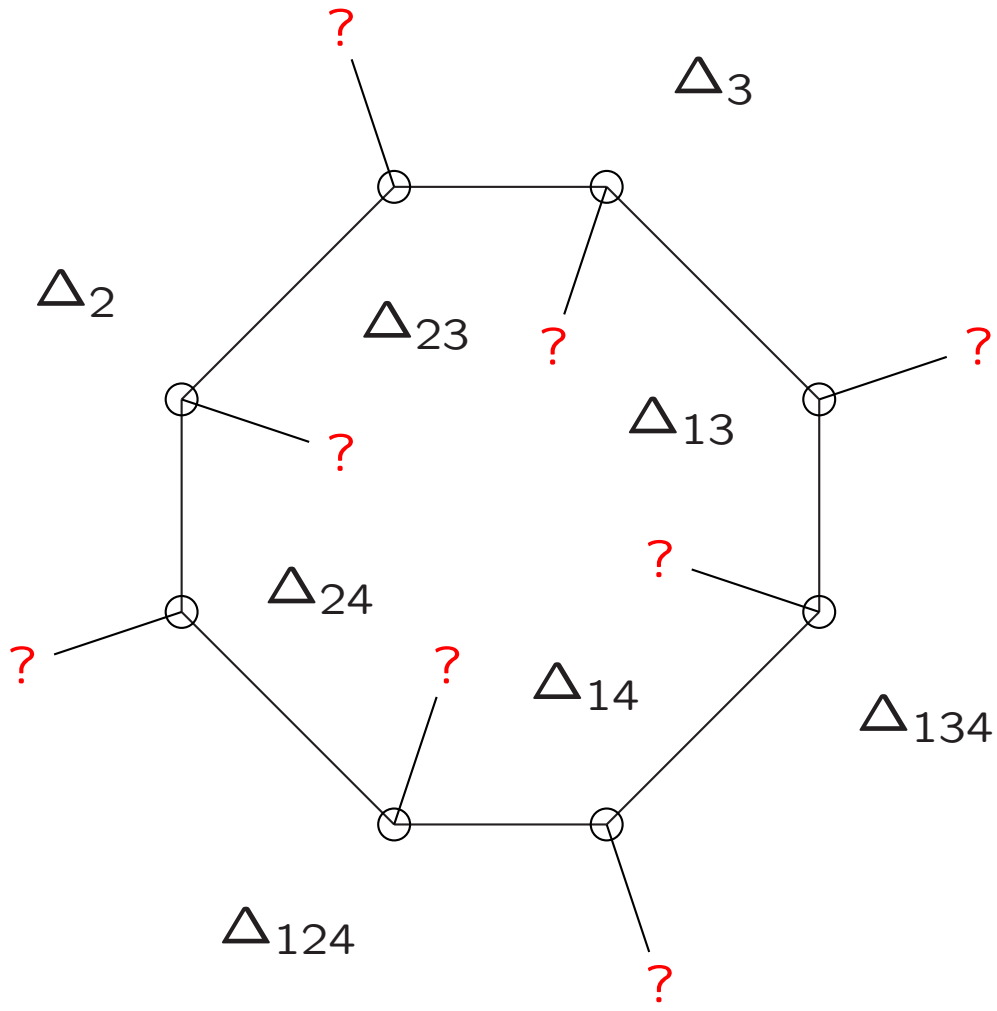
The rational expression $f = \frac{ac+bd}{e}$ is *subtraction-free*.

**Theorem 1** *If the elements of a particular extended cluster evaluate positively at a given point, then so do all flag minors.*

## Main features of a general cluster algebra setup
(illustrated by the prototypical example)

- a family of generators of the algebra (the flag minors);
- a finite subset of "frozen" generators;
- grouping of remaining generators into overlapping "clusters;"
- combinatorial data accompanying each cluster (a pseudoline arrangement);
- "exchange relations" that can be written using those data;
- a "mutation rule" for producing new combinatorial data from the given one (braid moves).

# The missing mutations

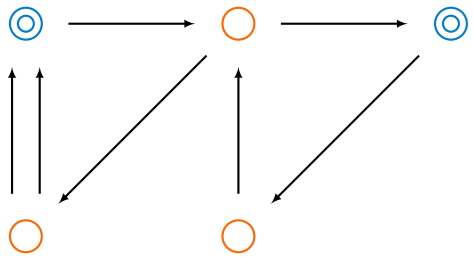


## Quivers

A *quiver* is a finite oriented graph.

Multiple edges are allowed.

Oriented cycles of length 1 or 2 are forbidden.



Two types of vertices: "frozen" and "mutable."

Ignore edges connecting frozen vertices to each other.

# Quiver mutations

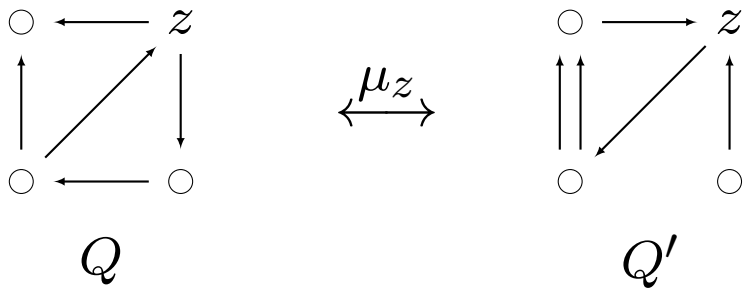
Quiver analogues of braid moves.

*Quiver mutation* $\mu_z : Q \mapsto Q'$ is computed in three steps.

**Step 1**. For each instance of $x \to z \to y$, introduce an edge $x \to y$.

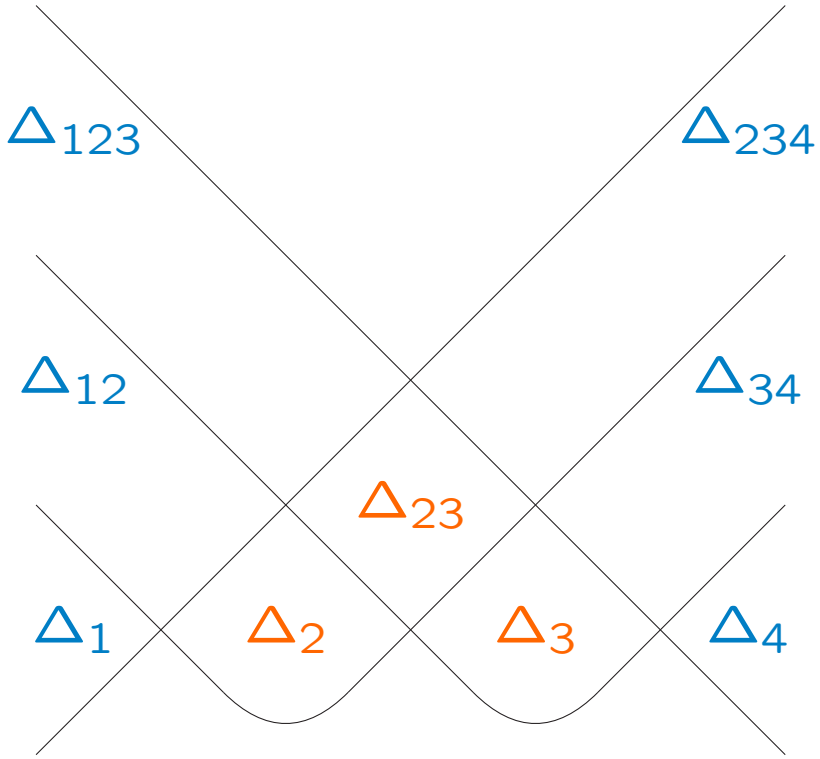
**Step 2**. Reverse the direction of all edges incident to $z$.

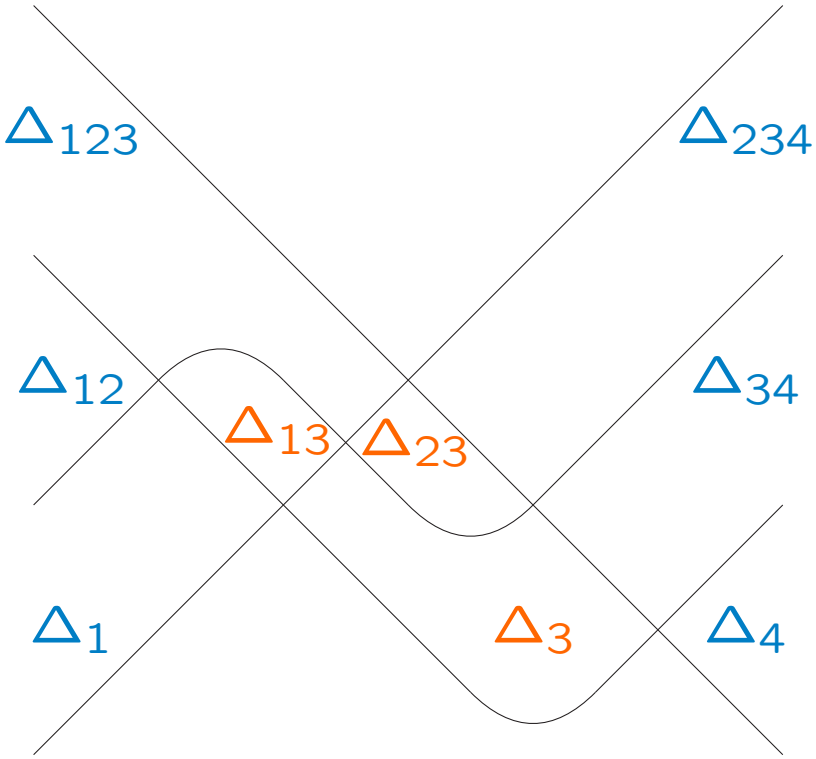
**Step 3**. Remove oriented 2-cycles.

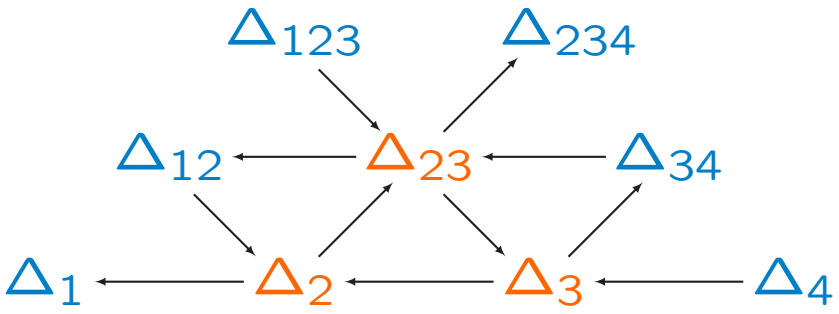


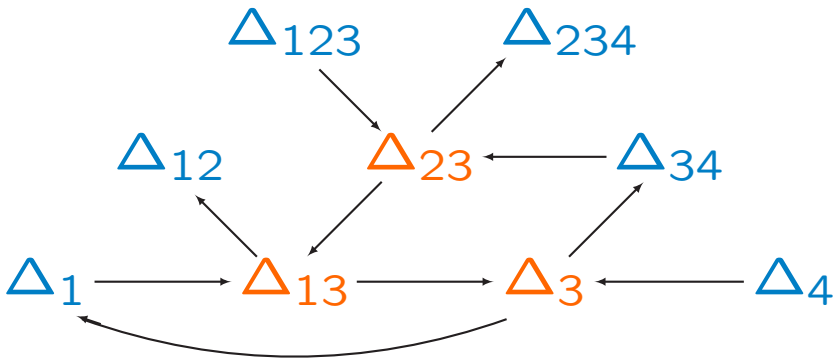

Easy: mutation of $Q'$ at $z$ recovers $Q$.

# Braid moves as quiver mutations



## Seeds

Let $\mathcal{F}$ be a field containing $\mathbb{C}$.

A *seed* in $\mathcal{F}$ is a pair $(Q, \mathbf{z})$ consisting of

- a quiver $Q$;
- an *extended cluster* $\mathbf{z}$, a tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$, labeled by the vertices of $Q$.

*coefficient variables* $\longleftrightarrow$ frozen vertices

*cluster variables* $\longleftrightarrow$ mutable vertices

## Clusters

cluster $=$ {cluster variables}

extended cluster $=$ {coefficient variables, cluster variables}

## Seed mutations

A *seed mutation* $\mu_z : (Q, \mathbf{z}) \mapsto (Q', \mathbf{z}')$ is defined by

- $\mathbf{z}' = \mathbf{z} \cup \{z'\} \setminus \{z\}$ where

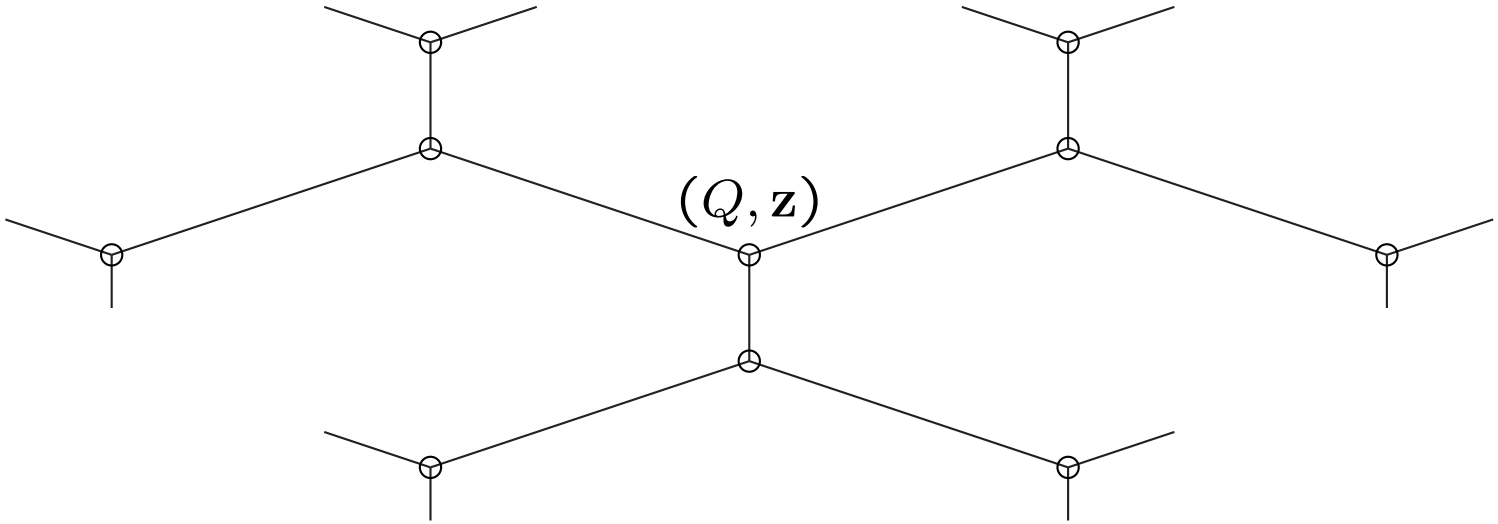
$$z\, z' = \prod_{z \leftarrow y} y + \prod_{z \rightarrow y} y$$

  (the *exchange relation*);

- $Q' = \mu_z(Q)$.

## Definition of a cluster algebra $\mathcal{A}(Q, \mathbf{z})$ (restricted generality)

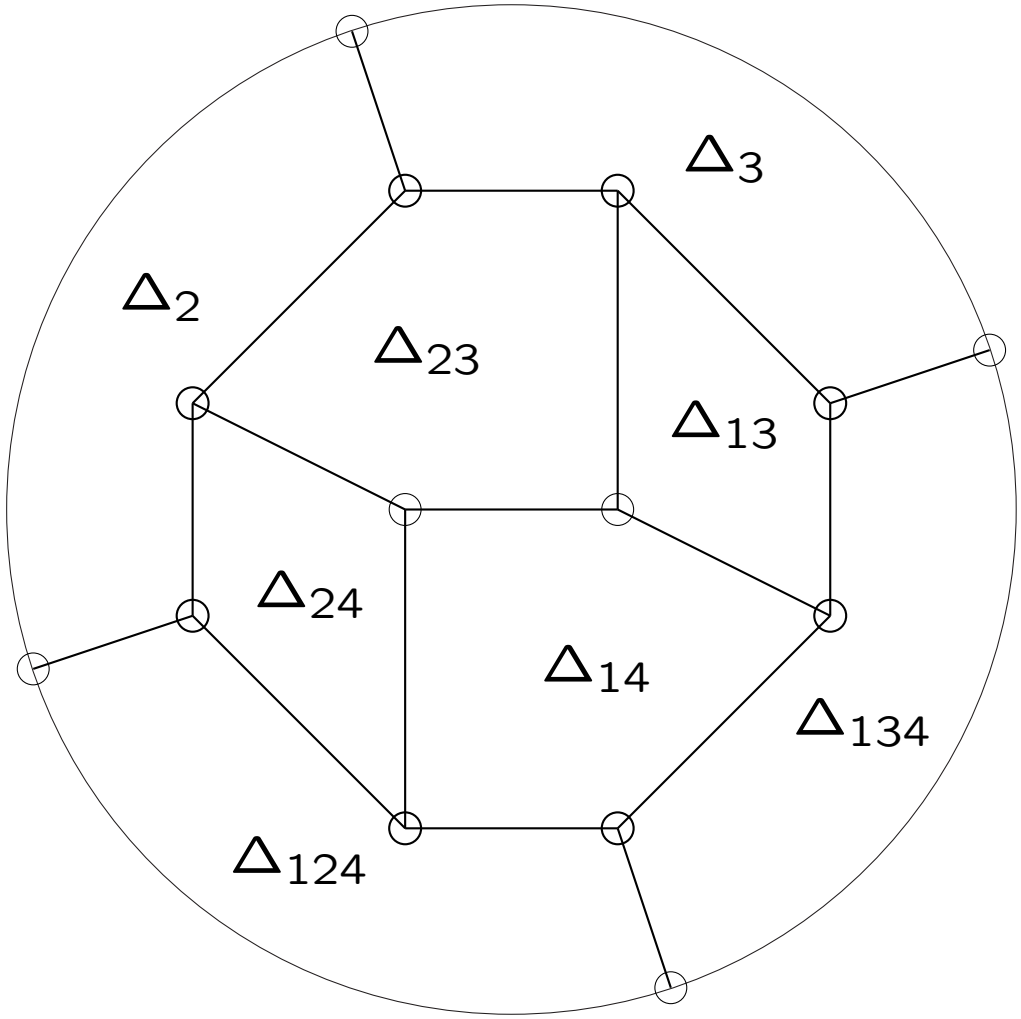
$\mathcal{A}(Q, \mathbf{z})$ is generated inside $\mathcal{F}$ by the union of all extended clusters obtained from the initial seed $(Q, \mathbf{z})$ by iterated mutations.




$\mathcal{A}(Q, \mathbf{z})$ is determined (up to isomorphism) by the mutation equivalence class of the quiver $Q$.

For now: cluster algebras of *geometric type* with *skew-symmetric* exchange matrices.

## Cluster combinatorics for $\mathbb{C}[\mathrm{SL}_4]^N$




$$\Omega = -\Delta_1\Delta_{234} + \Delta_2\Delta_{134}$$

## Examples of cluster algebras

**Theorem 2** [J. Scott, *Proc. London Math. Soc.* **92** (2006)] *The homogeneous coordinate ring of any Grassmannian* $\mathrm{Gr}_{k,r}(\mathbb{C})$ *has a natural cluster algebra structure.*

**Theorem 3** [C. Geiss, B. Leclerc, and J. Schröer, *Ann. Inst. Fourier* **58** (2008)] *The coordinate ring of any partial flag variety* $\mathrm{SL}_m(\mathbb{C})/P$ *has a natural cluster algebra structure.*

This can be used to build a cluster structure in each ring $\mathbb{C}[\mathrm{SL}_m]^N$.

Other examples: coordinate rings of $G/P$'s, double Bruhat cells, Schubert varieties, etc.

## Cluster structures in commutative rings

How can one show that a ring $R$ is a cluster algebra?

$R$ has to be an integral domain, and a $\mathbb{C}$-algebra.

Can use $\mathcal{F} = \mathrm{QF}(R)$ (quotient field).

Challenge: find a seed $(Q, \mathbf{z})$ in $\mathrm{QF}(R)$ such that $\mathcal{A}(Q, \mathbf{z}) = R$.

## Proving that a ring is a cluster algebra

$$\text{star}(Q, \mathbf{z}) \stackrel{\text{def}}{=} \mathbf{z} \cup \{\text{cluster variables from adjacent seeds}\}$$

**Proposition 4** *Let $R$ be a finitely generated $\mathbb{C}$-algebra and a normal domain. If all elements of* $\text{star}(Q, \mathbf{z})$ *belong to $R$ and are pairwise coprime, then $R \supset \mathcal{A}(Q, \mathbf{z})$.*

*If, in addition, $R$ has a set of generators each of which appears in the seeds mutation-equivalent to $(Q, \mathbf{z})$, then $R = \mathcal{A}(Q, \mathbf{z})$.*

## What do we gain from a cluster structure?

**1.** A sensible notion of (total) positivity.

**2.** A "canonical" basis, or a part of it.

**3.** A uniform perspective and general tools of cluster theory.

# PART 2: BASIC STRUCTURAL RESULTS

## The Laurent phenomenon

**Theorem 5** *All cluster variables are Laurent polynomials in the elements of the initial extended cluster.*

**Exercise.** Suppose that a sequence $x_0, x_1, x_2, \ldots$ satisfies

$$x_n x_{n+5} = x_{n+1} x_{n+4} + x_{n+2} x_{n+3}\,.$$

Interpret this recurrence as a special case of cluster mutation. Conclude that each $x_n$ is a Laurent polynomial in $x_0, \ldots, x_4$.

Setting $x_0 = \cdots = x_4 = 1$ produces a sequence of integers

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, 165713, \ldots$$

## More on the Laurent phenomenon

**Corollary 6** *Any element of a cluster algebra, when expressed in terms of a fixed extended cluster, is a Laurent polynomial.*

**Theorem 7** *Coefficient variables do not appear in the denominators of these Laurent polynomials.*

**Conjecture 8** (Laurent Positivity) *When expressed in terms of an arbitrary extended cluster, each cluster variable is given by a Laurent polynomial with positive coefficients.*

Special cases: P. Caldero–M. Reineke, G. Musiker–R. Schiffler-L. Williams, H. Nakajima, R. Kedem–P. Di Francesco, *et al.*

## Cluster monomials

A *cluster monomial* is a product of (powers of) elements of the same extended cluster.

In $\mathcal{A} = \mathbb{C}[\mathsf{SL}_4/N]$, the cluster monomials form a linear basis. This is an example of Lusztig's *dual canonical basis*.

**Theorem 9** [G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon, `arXiv:1203.1307`]. *In a cluster algebra defined by a quiver, the cluster monomials are linearly independent.*

Cluster monomials do *not* form a linear basis unless the number of clusters is finite.

## Positivity conjectures

**Conjecture 10** (Strong Positivity Conjecture) *Any cluster algebra has an additive basis which*

- *includes the cluster monomials and*
- *has nonnegative structure constants.*

The Strong Positivity Conjecture implies Laurent Positivity.

Conjecture 10 suggests the existence of a *monoidal categorification* [B. Leclerc–D. Hernandez, H. Nakajima].

## Mutation-acyclic quivers

A quiver is called *mutation-acyclic* if it can be transformed by iterated mutations into a quiver whose mutable part is acyclic.

**Theorem 11** [A. Buan, R. Marsh, and I. Reiten, *Comment. Math. Helv.* **83** (2008)] *A full subquiver of a mutation-acyclic quiver is mutation-acyclic.*

**Theorem 12** [Y. Kimura and F. Qin, `arXiv:1205.2066`]
*The strong positivity conjecture holds for any cluster algebra defined by a mutation-acyclic quiver.*

## Cluster complex and exchange graph

**Theorem 13** [M. Gekhtman, M. Shapiro, and A. Vainshtein, *Math. Res. Lett.* **15** (2008)] *In a given cluster algebra, each seed is determined by its cluster. Two seeds are adjacent if and only if their clusters share all elements but one.*

The combinatorics of clusters and exchanges is encoded by the *cluster complex*. This is a simplicial complex in which:

$$\begin{array}{rcl} \text{vertices} & \longleftrightarrow & \text{cluster variables} \\ \text{maximal simplices} & \longleftrightarrow & \text{clusters} \end{array}$$

By Theorem 13, the cluster complex is a *pseudomanifold*. Its dual graph is the *exchange graph* of the cluster algebra:

$$\begin{array}{rcl} \text{vertices} & \longleftrightarrow & \text{seeds/clusters} \\ \text{edges} & \longleftrightarrow & \text{mutations} \end{array}$$

## Cluster type

The mutable part of $Q$ determines the *(cluster) type* of $\mathcal{A}(Q, \mathbf{z})$.

Example: $\mathsf{SL}_4/N$ and $\mathsf{Gr}_{2,6}$.

**Conjecture 14** *The cluster complex depends only on the type of a cluster algebra.*

**Theorem 15** [G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon, `arXiv:1203.1307`]. *The exchange graph depends only on the type of a cluster algebra.*

## Cluster algebras of finite type

A cluster algebra is of *finite type* if it has finitely many seeds (equivalently, finitely many cluster variables).

This property turns out to depend only on the cluster *type*.

Next: classify all cluster algebras of finite type; describe their underlying combinatorics.

## Example: Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$

A point in $\mathrm{Gr}_{2,n+3}(\mathbb{C})$ is represented by a $2 \times (n+3)$ matrix:

$$z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1,n+3} \\ z_{21} & z_{22} & \cdots & z_{2,n+3} \end{bmatrix}$$

The homogeneous coordinate ring of $\mathrm{Gr}_{2,n+3}(\mathbb{C})$ (with respect to its Plücker embedding) is generated by the *Plücker coordinates*

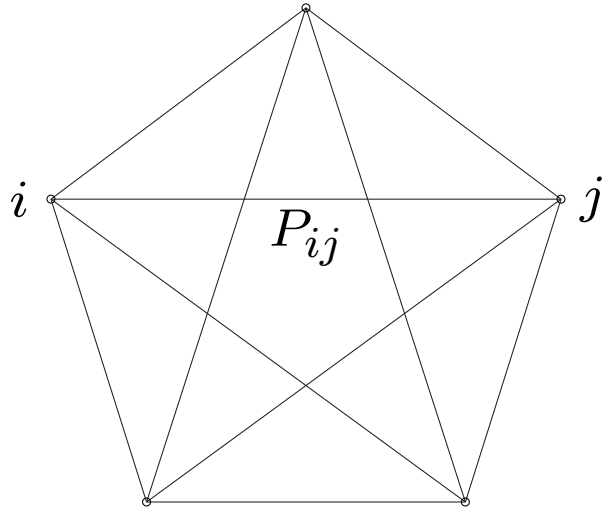
$$P_{ij} = \det \begin{bmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{bmatrix}.$$

They satisfy the *Grassmann-Plücker relations*

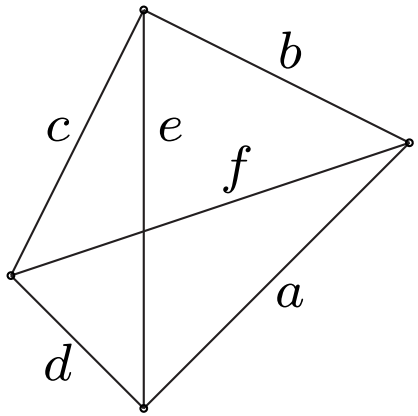
$$P_{ik}\, P_{jl} = P_{ij}\, P_{kl} + P_{il}\, P_{jk} \quad (i < j < k < l).$$

# Ptolemy relations

Plücker coordinate $\longleftrightarrow$ side/diagonal of a convex $(n+3)$-gon:

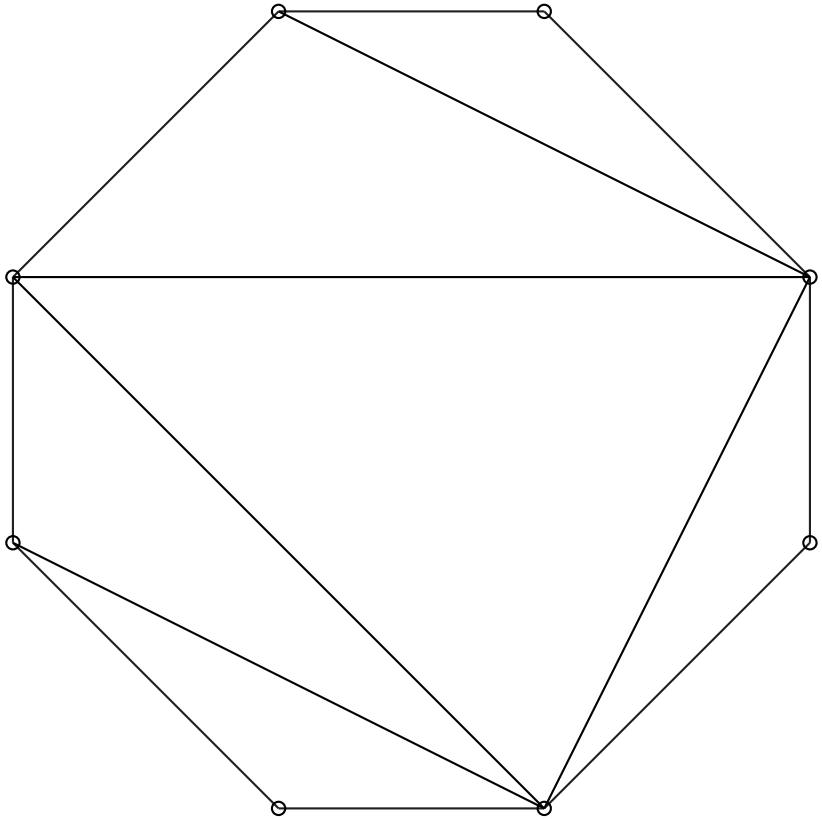



Grassmann-Plücker relation $\longleftrightarrow$ *Ptolemy relation*:



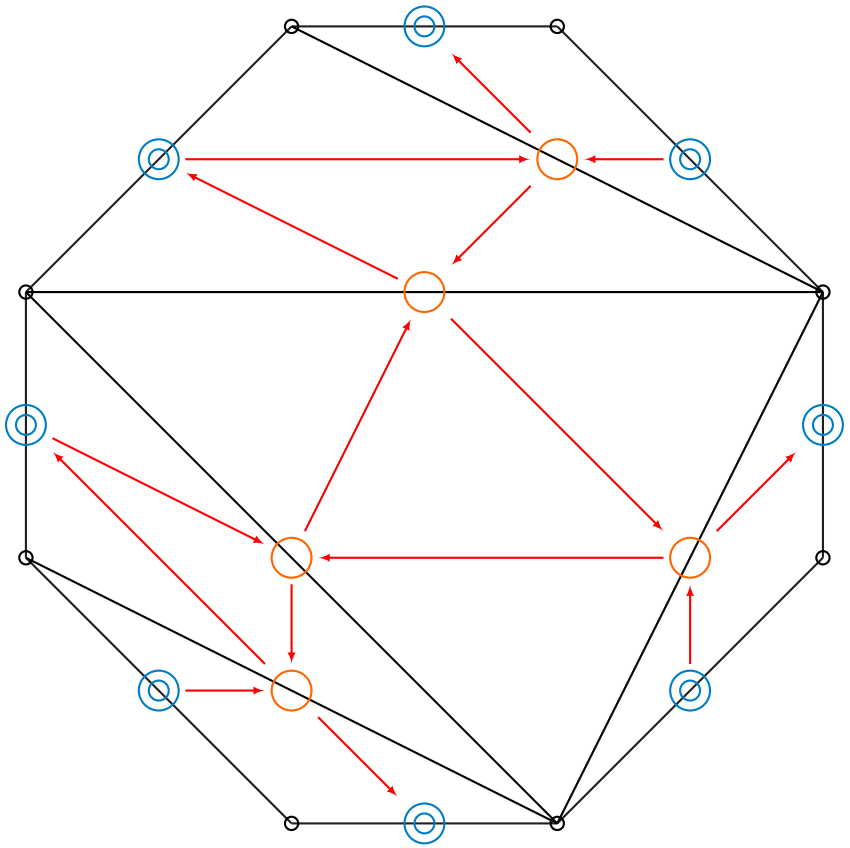

$$ef = ac + bd$$

## Cluster structure on a Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$



| | | |
|---|---|---|
| cluster variables | $\longleftrightarrow$ | diagonals |
| frozen variables | $\longleftrightarrow$ | sides |
| clusters/seeds | $\longleftrightarrow$ | triangulations |
| mutations | $\longleftrightarrow$ | flips |
| exchange relations | $\longleftrightarrow$ | Grassmann-Plücker relations |

# Cluster structure on a Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$



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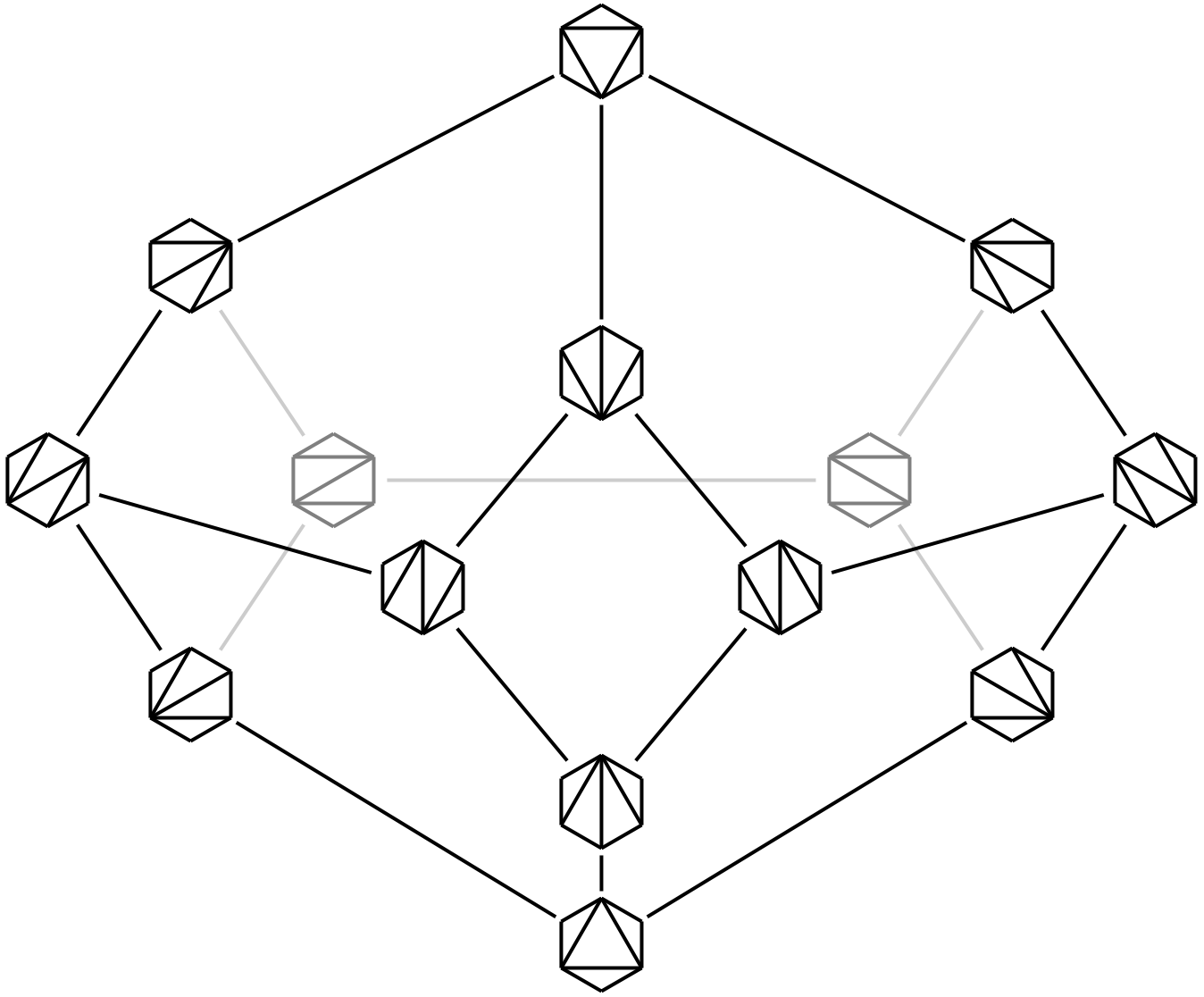
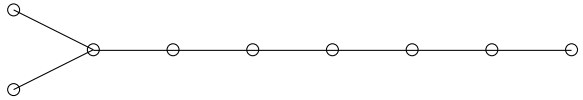
## Exchange graph for $\mathbb{C}[\mathrm{Gr}_{2,n+3}]$: the associahedron



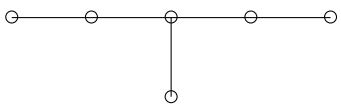
## Finite type classification

$A_n$ $(n \geq 1)$

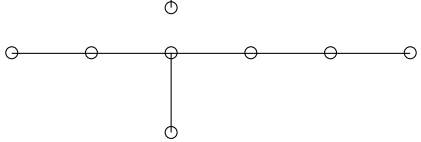
$D_n$ $(n \geq 4)$



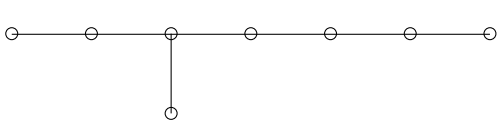
$E_6$



$E_7$



$E_8$



**Theorem 16** *A cluster algebra is of finite type if and only if the mutable part of its quiver at some seed is an orientation of a (simply-laced) Dynkin diagram.*

The type of this Dynkin diagram in the Cartan-Killing nomenclature is uniquely determined by the cluster algebra.

## Cluster types of some coordinate rings

| | |
|---|---|
| $\mathbb{C}[\mathsf{Gr}_{2,n+3}]$ | $A_n$ |
| $\mathbb{C}[\mathsf{Gr}_{3,6}]$ | $D_4$ |
| $\mathbb{C}[\mathsf{Gr}_{3,7}]$ | $E_6$ |
| $\mathbb{C}[\mathsf{Gr}_{3,8}]$ | $E_8$ |
| $\mathbb{C}[\mathsf{SL}_3]^N$ | $A_1$ |
| $\mathbb{C}[\mathsf{SL}_4]^N$ | $A_3$ |
| $\mathbb{C}[\mathsf{SL}_5]^N$ | $D_6$ |

## Cluster complexes in finite type

**Theorem 17** [F. Chapoton, S.F., A. Zelevinsky, *Canad. Math. Bull.* **45** (2002)] *The cluster complex of a cluster algebra of finite type is the dual simplicial complex of a simple convex polytope.*

These polytopes are called *generalized associahedra*.

Type $A_n$: ordinary associahedron (Stasheff's polytope).

## Almost positive roots

$\mathcal{A}$ cluster algebra of finite type

$\Phi$ the corresponding crystallographic root system

**Theorem 18** *The cluster variables in $\mathcal{A}$ are in bijection with the roots in $\Phi$ which are either positive or negative simple.*

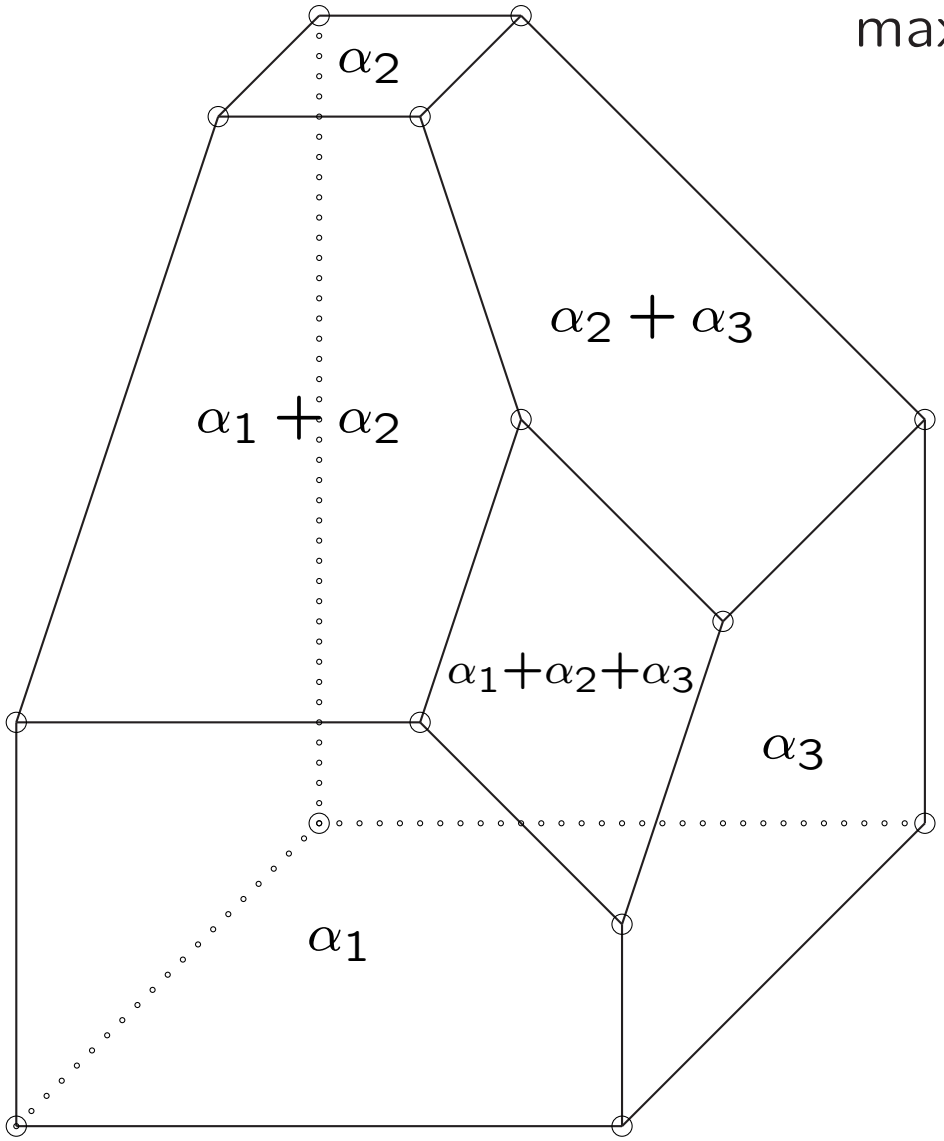
$$\text{cluster variable } x \longleftrightarrow \text{root } \alpha$$
$$\text{denominator of } x \longleftrightarrow \text{simple root expansion of } \alpha$$

The combinatorics of the cluster complex and the geometry of generalized associahedra can be described in root-theoretic terms.

# Polyhedral realization of the associahedron of type $A_3$

$$\max(-z_1\,,\,-z_3\,,\,z_1\,,\,z_3\,,\,z_1+z_2\,,\,z_2+z_3) \le 3/2$$

$$\max(-z_2\,,\,z_2\,,\,z_1+z_2+z_3) \le 2$$



## Enumerative results

**Theorem 19** *The number of clusters in a cluster algebra of finite type is equal to*

$$N(\Phi) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1},$$

*where* $e_1, \ldots, e_n$ *are the exponents, and* $h$ *is the Coxeter number.*

$N(\Phi)$ is the *Catalan number* associated with the root system $\Phi$.

| $\Phi$ | $A_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|
| $N(\Phi)$ | $\frac{1}{n+2}\binom{2n+2}{n+1}$ | $\frac{3n-2}{n}\binom{2n-2}{n-1}$ | 833 | 4160 | 25080 |

## Catalan combinatorics of arbitrary type

Besides clusters, the numbers $N(\Phi)$ enumerate:

- $ad$-*nilpotent ideals* in a Borel subalgebra of a semisimple Lie algebra;
- antichains in the *root poset*;
- regions of the *Catalan arrangement* inside the fundamental chamber;
- orbits of the Weyl group action on the quotient $Q/(h+1)Q$ of the root lattice;
- conjugacy classes of elements $x$ of a semisimple Lie group which satisfy $x^{h+1} = 1$;
- *non-crossing partitions* of the appropriate type.

# PART 3: PERIODICITY AND GRASSMANNIANS

## The pentagon recurrence

Let $f_1 = x$, $f_2 = y$, and

$$f_{n+1} = \frac{f_n + 1}{f_{n-1}}.$$

We get:

$$x,\ y,\ \frac{y+1}{x},\ \frac{x+y+1}{xy},\ \frac{x+1}{y}.$$

Next: $x$ and $y$.

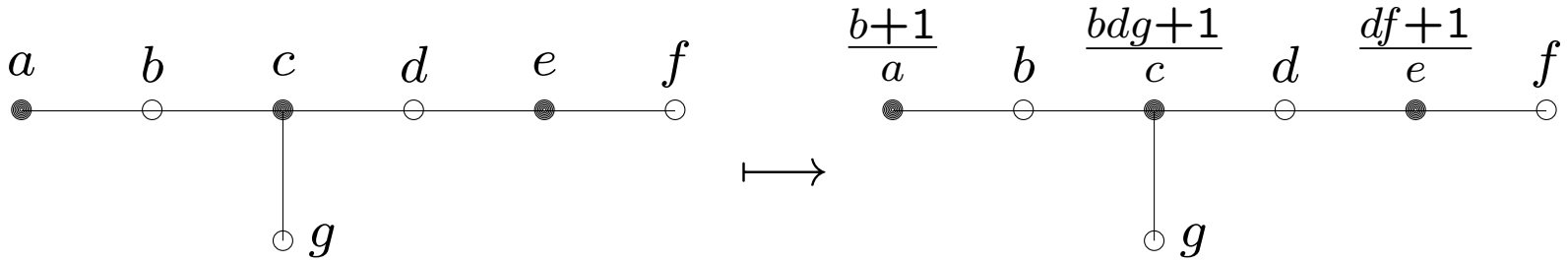
Explanation: iterated cluster mutations in type $A_2$.

## Cluster transformations on bipartite graphs

bipartite graph $\longleftrightarrow$ quiver with each vertex a source or a sink

Mutations at vertices of the same color commute.

Bipartite mutation dynamics: all black, then all white.




The quiver does not change.

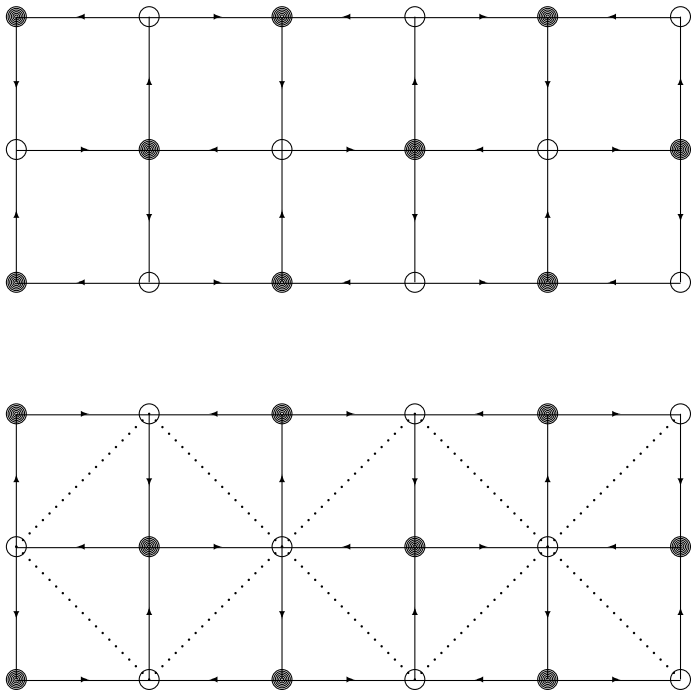
## Zamolodchikov periodicity conjecture

**Theorem 20** *Bipartite mutation dynamics on a simply-laced Dynkin diagram has period $h$+2 where $h$ is the Coxeter number.*

**Theorem 21** *Bipartite mutation dynamics is periodic if and only if the underlying graph is a simply-laced Dynkin diagram.*

Recall: exchange graph depends only on the cluster type.

## Cluster transformations for pairs of Dynkin diagrams



The *octahedron recurrence* (a.k.a. Hirota's equation).

## Zamolodchikov periodicity for pairs of Dynkin diagrams

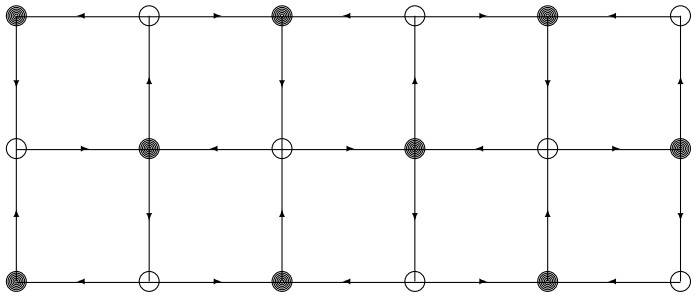
Conjecture [F. Ravanini-A. Valleriani-R. Tateo and A. Kuniba-T. Nakanishi, $\approx$ 1993]:

**Theorem 22** [B. Keller, *Ann. Math.*. **177** (2013)] *Bipartite mutation dynamics for pairs of Dynkin diagrams is periodic, with the period dividing the sum of the two Coxeter numbers.*

Analogues for non-simply-laced types were obtained by R. Inoue, O. Iyama, B. Keller, A. Kuniba, T. Nakanishi, and J. Suzuki.

## Volkov's proof of periodicity for type $A_{n-1} \times A_{m-1}$

Adaptation of [A. Volkov, *Comm. Math. Phys.* **276** (2007)].



Idea: interpret cluster transformations on $A_{n-1} \times A_{m-1}$ using the cluster structure on the appropriate "Plücker ring."

## Plücker coordinates

$\mathcal{A}_{n,n+m}$ = homogeneous coordinate ring of $\mathrm{Gr}_{n,n+m}$

$\mathcal{A}_{n,n+m} = \{\mathrm{SL}_n\text{-invariants of } (n+m)\text{-tuples } v_1, \ldots, v_{n+m} \in \mathbb{C}^n\}$

$I = \{i_1 < \cdots < i_n\} \subset \{1, \ldots, n+m\}$

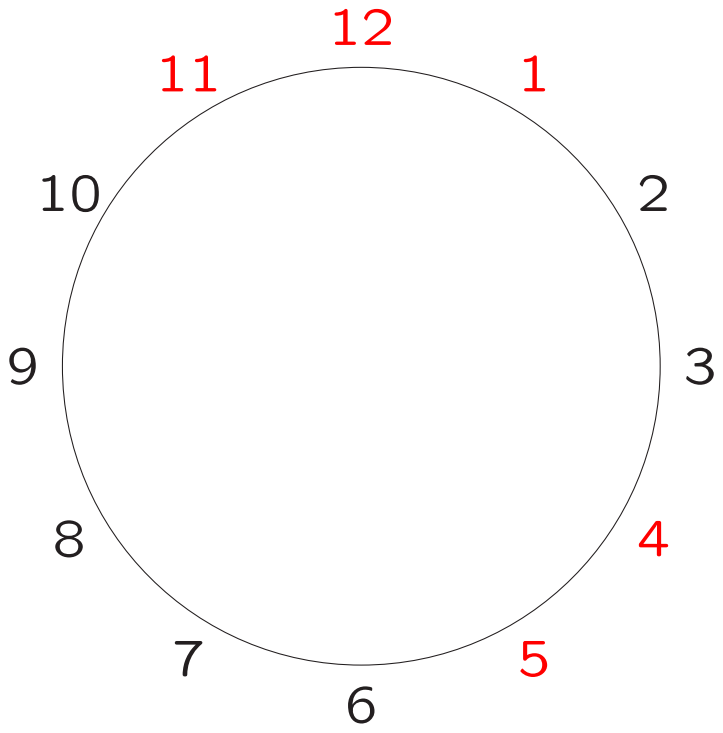
*Plücker coordinate*: $[I] = [i_1, \ldots, i_n]$ (formerly denoted by $P_I$)

$[I]$ is an $n \times n$ minor of a generic $n \times (n+m)$ matrix.

**Theorem 23** [First Fundamental Theorem of Invariant Theory, H. Weyl, 1930s] *$\mathcal{A}_{n,n+m}$ is generated by the Plücker coordinates.*

## Special Plücker coordinates

These are Plücker coordinates $[I]$ where $I$ consists of one or two "contiguous segments" modulo $n+m$.




$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

Coefficient variables: $[I]$, with $I$ a single segment

Grading by $\mathbb{Z}/(n+m)\mathbb{Z}$: $\ell(I) = 4 + 11 \pmod{12}$.

# Three-term relations for special Plücker coordinates

$$\begin{array}{c} \begin{array}{cccccccccccccc} & & a & & & b & & & & c & & & & d & \\ \end{array} \\ \left[\begin{array}{cc|c|cc|c|ccc|c|ccc|c|c} * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \end{array}\right] \end{array}$$

$$J = [a+1, b-1] \cup [c+1, d-1]$$

$$[Jac][Jbd] = [Jab][Jcd] + [Jad][Jbc]$$

## Special seeds in a Grassmannian

cluster = {non-coefficient special $[I]$ with $\ell(I)=k$ or $\ell(I)=k+1$}

$$\boxed{n=3 \qquad m=4 \qquad k=6}$$

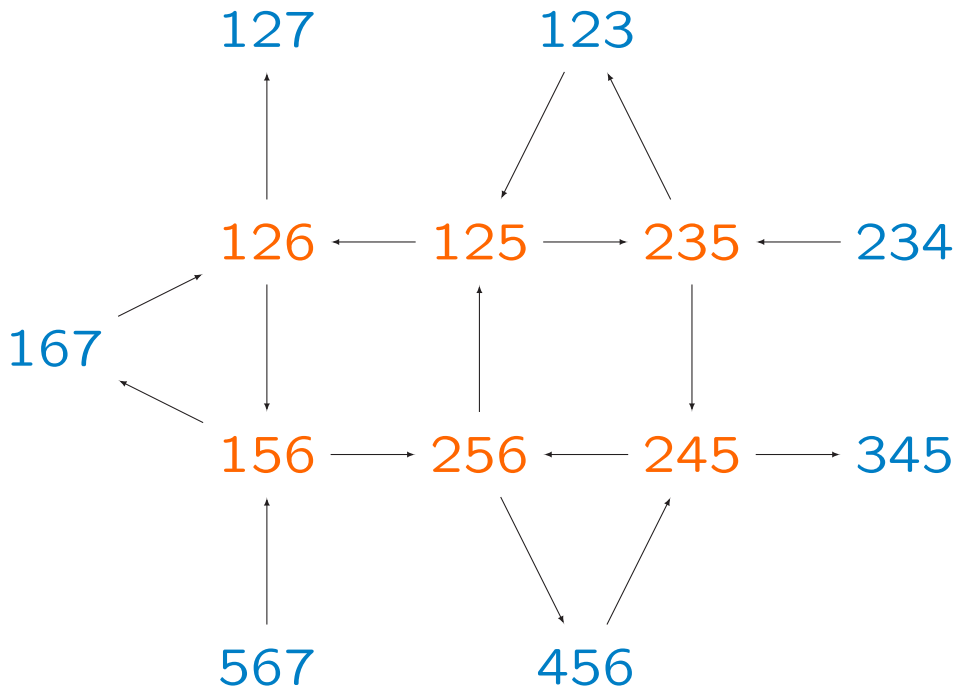
coefficients = $\{[123],[234],[345],[456],[567],[167],[127]\}$

cluster = $\{[125],[156],[245],[126],[256],[235]\}$

exchange relations:

$$[125][237] = [123][256] + [126][235]$$
$$[156][267] = [126][567] + [167][256]$$
$$[245][356] = [235][456] + [256][345]$$
$$[126][157] = [167][125] + [127][156]$$
$$[256][145] = [156][245] + [125][456]$$
$$[235][124] = [125][234] + [123][245]$$

## Quiver for a special seed



## Leclerc-Zelevinsky conjectures

$I$ and $J$ are called *weakly separated*

$$\Updownarrow$$

$I \setminus J$ and $J \setminus I$ are non-crossing

**Theorem 24**

[V. Danilov, A. Karzanov, and G. Koshevoy, *J. Algebraic Combin.* **32** (2010), S. Oh, A. Postnikov, and D. Speyer, `arXiv:1109.4434`]
*All maximal collections of pairwise weakly separated $n$-element subsets of the set $\{1, 2, \ldots, n+m\}$ have the same cardinality. These maximal collections correspond precisely to the clusters in $\mathcal{A}_{n,n+m}$ which consist entirely of Plücker coordinates.*

## Cluster structures in Grassmannians: further directions

- Explicit description of cluster variables
- Compatibility criteria
- Cluster structures in classical rings of invariants
- Additive bases (dual canonical, etc.)
- Totally nonnegative Grassmannians
- Positroid varieties
- $\mathrm{SL}_n$ local systems on Riemann surfaces
- Other partial flag varieties
- Other Lie types
- Connections with Poisson geometry, integrable systems, etc.

# PART 4: CLUSTER ALGEBRAS IN FULL GENERALITY

## Three levels of mutation dynamics

| mutable part of a quiver | skew-symmetrizable matrix |
|---|---|
| edges to/from frozen vertices | $Y$-variables |
| cluster variables | cluster variables |

## Skew-symmetrizable matrices

An $n \times n$ integer matrix $B = (b_{ij})$ is *skew-symmetrizable* if

$$d_i \, b_{ij} = -d_j \, b_{ji}$$

for some positive integers $d_1, \ldots, d_n$.

Skew-symmetric matrices $\longleftrightarrow$ quivers.

*Matrix mutation*: $\mu_k(B) = B' = (b'_{ij})$ where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(b_{ik} b_{kj}, 0) & \text{otherwise.} \end{cases}$$

## $Y$-seeds

*$Y$-seed*: pair $(B, \mathbf{y})$ where

- $B = (b_{ij})$ is an $n \times n$ skew-symmetrizable matrix;
- $\mathbf{y} = (y_1, \ldots, y_n)$ is an $n$-tuple of elements of a *semifield* $(\mathcal{P}, \oplus, \cdot)$.

*$Y$-seed mutation*:

$$B \mapsto B' = \mu_k(B)$$

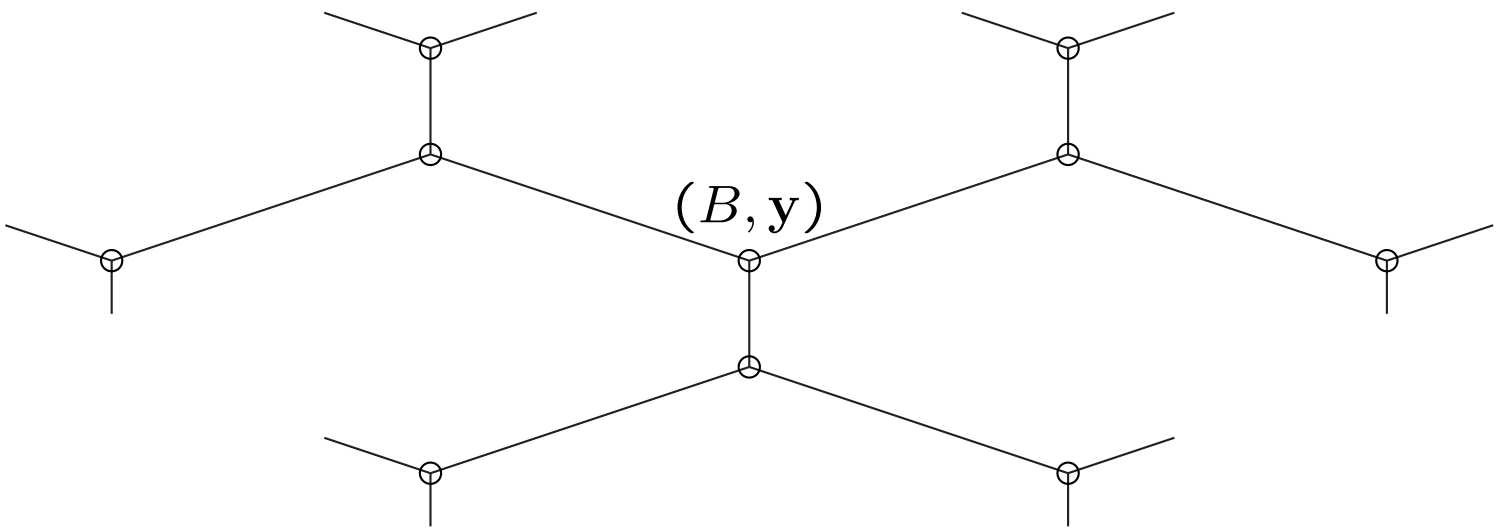
$$\mathbf{y} \mapsto \mathbf{y}' = (y'_1, \ldots, y'_n)$$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j \, y_k^{\max(b_{kj},0)} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k. \end{cases}$$

*Tropical semifield*: multiplicative generators $u_1, u_2, \ldots$, with

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)} .$$

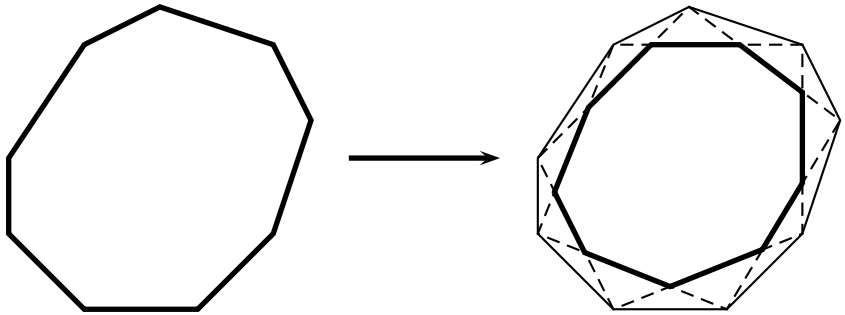
# $Y$-patterns

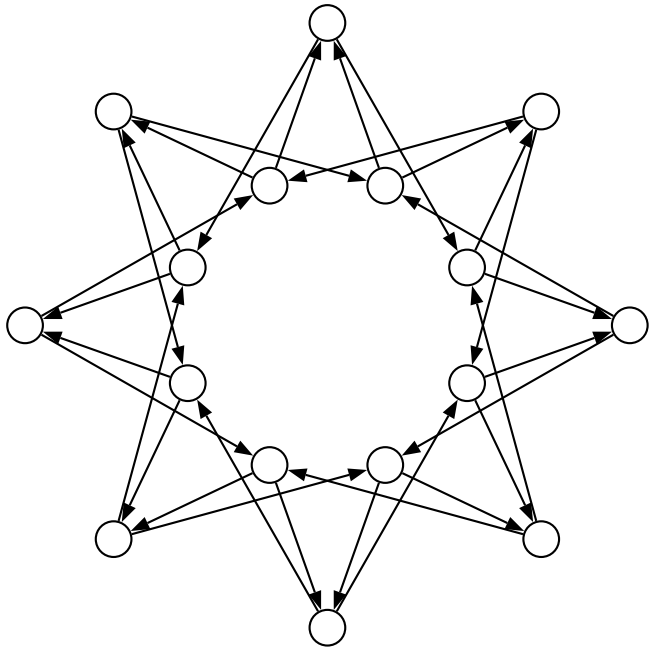



- Discrete integrable systems (*$Y$-systems*) in theoretical physics
- Transformations of *shear coordinates* under flips
- *Fock-Goncharov varieties* ("cluster $X$-varieties")
- *Wall-crossing formulas* in Donaldson-Thomas/string theory [M. Kontsevich-Y. Soibelman, D. Gaiotto-G. Moore-A. Neitzke]
- *The pentagram map* and its generalizations

# The pentagram map

[R. Schwartz 1992, M. Glick 2010]





## Seeds

Use an ambient field $\mathcal{F}$ containing $\mathbb{C}\mathcal{P}$.

*Seed*: triple $(B, \mathbf{y}, \mathbf{x})$ where

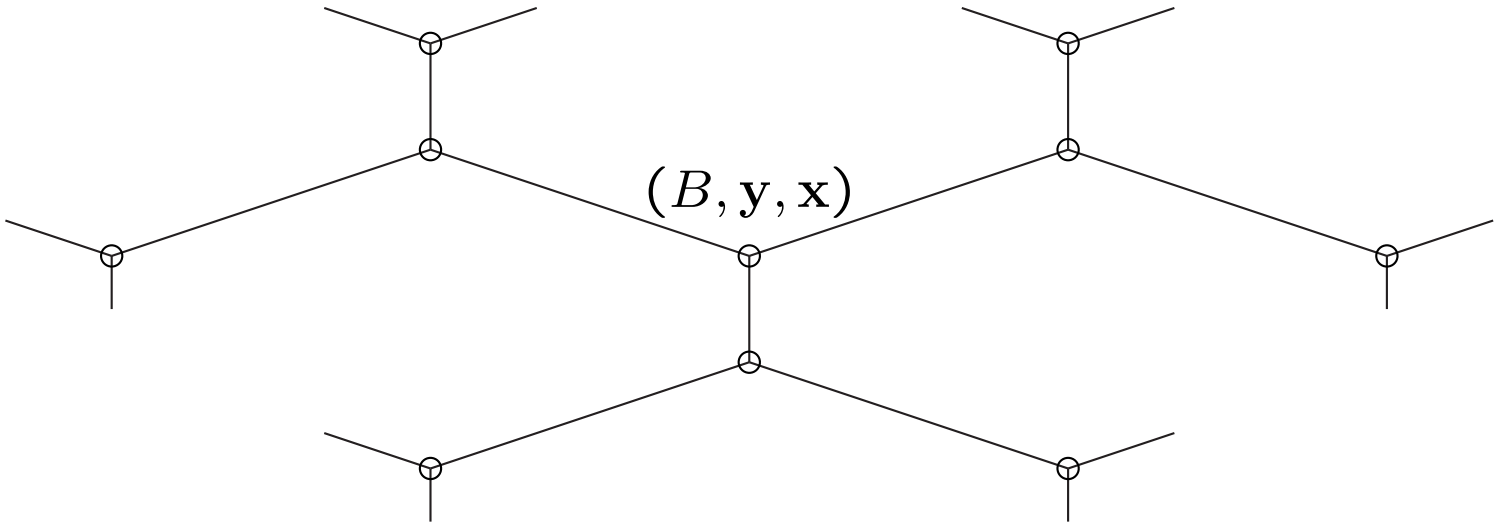
- $B = (b_{ij})$ is an $n \times n$ skew-symmetrizable matrix;
- $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{P}^n$.
- $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{F}^n$.

*Seed mutations*: use the exchange relation

$$x_k \, x_k' = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik}>0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik}<0} x_i^{-b_{ik}} \, .$$

## Cluster algebras over an arbitrary semifield

The (normalized) *cluster algebra* $\mathcal{A}(B, \mathbf{y}, \mathbf{x})$ is generated (over some ring containing $\mathbb{Z}\mathcal{P}$) by all cluster variables in all seeds obtained from $(B, \mathbf{y}, \mathbf{x})$ by iterated mutations.



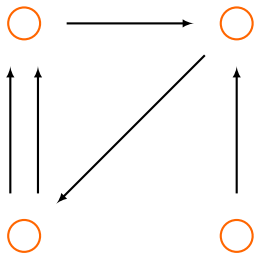
## Extending cluster theory into full generality

- The Laurent phenomenon
- Skew-symmetrizable cluster algebras of geometric type
- Examples from Lie theory
- Notion of cluster type. Finite type classification
- Generalized associahedra
- Folding
- Zamolodchikov periodicity

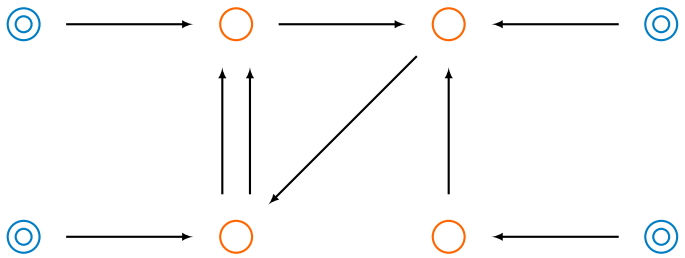
## Exchange graphs

**Conjecture 25** *The exchange graph for a $Y$-pattern (resp., a cluster algebra) depends solely on the exchange matrix $B$. Moreover these two exchange graphs coincide with each other.*

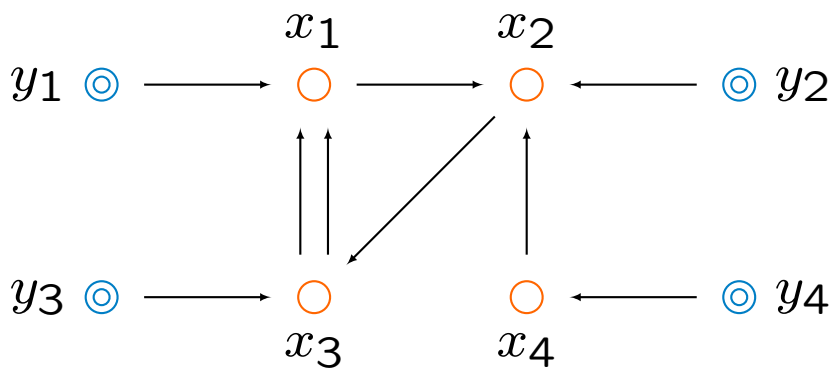
"Smallest" exchange graph: trivial coefficients.



"Largest" exchange graph: *principal coefficients*.



## Separation of additions




Each cluster variable in $\mathcal{A}_{\text{principal}}(B)$ is a subtraction-free rational expression (and a Laurent polynomial) in $x_1, x_2, \ldots, y_1, y_2, \ldots$.

**Theorem 26** *Let $x$ be a cluster variable in a cluster algebra $\mathcal{A}$ over a semifield $\mathcal{P}$, with ambient field $\mathcal{F}$ and initial exchange matrix $B$. Let $X$ be the subtraction-free rational expression for the corresponding cluster variable in $\mathcal{A}_{\text{principal}}(B)$. Then*

$$x = \frac{X|_{\mathcal{F}}(x_1, \ldots, x_n; y_1, \ldots, y_n)}{X|_{\mathcal{P}}(1, \ldots, 1; y_1, \ldots, y_n)}.$$

## $Y$-patterns via cluster dynamics

**Theorem 27** *Let $y'_j$ be a variable in a $Y$-pattern with initial data $(B, \mathbf{y})$, $\mathbf{y} = (y_1, \ldots, y_n)$. Then*

$$y'_j = Y'_j \prod_{i=1}^{n} X_i(1, \ldots, 1; y_1, \ldots, y_n)^{b'_{ij}}$$

*where*

- *$X_1, \ldots, X_n$ are the subtraction-free rational expressions for cluster variables in the corresponding seed for $\mathcal{A}_{\mathsf{principal}}(B)$,*
- *$Y'_j$ be the counterpart of $y_j$ in that seed;*
- *$B' = (b'_{ij})$ is the exchange matrix at that seed.*

## Post scriptum

**Conjecture 28** *Any cluster algebra (say defined by a quiver) is a free module over its subring generated by the coefficient variables.*

Example: the Plücker ring and its subring generated by the "cyclically contiguous" Plücker coordinates.