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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Karen Smith

Talk Title: Introduction to Frobenius splitting #1

Date: 08/27/12 Time: 2:00 am /pm (circle one)

List 6-12 key words for the talk: Frobenius splitting, Cohen-Macaulay, Hochster-Roberts

Please summarize the lecture in 5 or fewer sentances: The key idea of splitting is illustrated by proving the Hochster-Roberts theorem.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Karen Smith – Introduction to Frobenius splitting, F-regularity and test ideals

Motivation: a theorem this machinery can prove

Thm (Hochster – Roberts)
If G is a linearly reductive group acting algebraically on a smooth variety X, then X/G is Cohen–Macaulay.

Algebraic version of the Thm
G = linearly reductive group acting linearly on $S = k[x_1, \dots, x_n]$
then the ring of invariants
$S^G = \{ f \in S \mid g \cdot f = f \ \forall g \in G \} \subseteq S$ is Cohen–Macaulay.

Ex 1: $G = \{\pm 1\}$ acts on $\mathbb{R}^2$ by $(-1)\cdot(x,y) = (-x,-y)$
This induces an action on $\mathbb{R}[x,y]$ given by
$(-1)\cdot(x^a y^b) = (-x)^a \cdot (-y)^b = (-1)^{a+b} x^a y^b$
Hence $(\mathbb{R}[x,y])^G = \mathbb{R}[x^2, xy, y^2] \cong \dfrac{\mathbb{R}[s,t,u]}{(su - t^2)}$

Ex 2: $G = SL_2(\mathbb{C})$ acts on the variety $\mathbb{C}^{2\times n}$ = variety of $2\times n$ $\mathbb{C}$-matrices
$\Rightarrow G$ acts on $\mathbb{C}\begin{bmatrix} x_{11} & \dots & x_{1n} \\ x_{21} & \dots & x_{2n} \end{bmatrix} = S$ by left multiplication
$S^G = \mathbb{C}[2\times 2$ minors of the matrix above$]$ is the Plücker ring i.e. the homogeneous coordinate ring for the Plücker embedding of $G_2(\mathbb{C}^n)$.

Def: A local (graded) ring R is Cohen–Macaulay (CM) if it admits a regular sequence of length $d = \dim R$
($x_1, \dots, x_d \in R$ is a regular sequence if x_1 is a non zero-divisor and x_{i+1} is a non-zero-divisor on $R/(x_1, \dots, x_i)$)

If R is graded, take a Noether normalization i.e.
$$\underbrace{A = k[x_1, \dots, x_d]}_{\text{polynomial subalgebra}} \xhookrightarrow[\text{finite extension}]{} R$$
R CM $\Leftrightarrow$ R is free as an A-module

Def An algebraic group is *linearly reductive* if every finite dimensional representation splits into irreducible representations.

- in char 0
  - linearly reductive is the same as reductive
  - linearly reductive groups include GL_n, SL_n, tori, finite groups
- in char p
  - finite groups whose order is not divisible by p
  - tori i.e. $(k^*) \times \ldots \times (k^*)$

II SPLITTINGS

Consider $R \hookrightarrow S$ inclusion of rings. (this makes S an R-module)

Def: $R \hookrightarrow S$ *splits* if $\exists$ R-module map, $\varphi(1)=1$
$\varphi: S \to R$ which restricts to the identity on R
($\Rightarrow S \cong R \oplus S/R$ as R-modules)

Ex3: Let G be a finite group acting on a ring S. Assume $|G|$ is invertible in S, then $R = S^G \hookrightarrow S$ always splits.

Proof

$$S \longrightarrow R$$
$$s \longmapsto \frac{1}{|G|} \sum_{g \in G} g \cdot s \quad \text{is } R\text{-linear and gives a splitting}$$

Ex 3b: Similarly, if G is linearly reductive acting on $S = k[x_1, \ldots, x_n]$, then $S^G \hookrightarrow S$ splits.

General principle: if $R \hookrightarrow S$ is split and S is nice, then R is almost as nice.

IIA Frobenius splitting

R char $p > 0$

For simplicity assume R is a domain, R finitely generated over a perfect field k

e.g. $R = \dfrac{\mathbb{F}_p[x_1, \ldots, x_n]}{(\text{relations})}$

Def: The Frobenius map is the ring homomorphism
$$R \to R$$
$$r \mapsto r^p$$

$$\left[\begin{array}{l} R \text{ commutative} \Rightarrow (rs)^p = r^p s^p \\ (r+s)^p = r^p + \underbrace{\binom{p}{1} r^{p-1}s + \dots + \binom{p}{p-1}}_{\text{all zero}} rs^{p-1} + s^p = r^p + s^p \end{array}\right]$$

The image is a subring $\underset{\text{elements of } R \text{ that are } p^{th} \text{ powers}}{R^p \subseteq R}$

Def R is Frobenius split (F-split $\Leftrightarrow$ F-pure) if $R^p \hookrightarrow R$ splits in the category of R^p-modules.

Ex $R = \mathbb{F}_p[x,y]$ is F-split
$R^p = \mathbb{F}_p[x^p, y^p]$
$\{x^i y^j\}_{0 \le i, j \le p-1}$ is a free basis for $R/R^p \Rightarrow R \cong \bigoplus_{0 \le i,j \le p-1} R^p x^i y^j$
More generally, the coordinate ring of any smooth affine variety over a perfect field of characteristic p is F-split.

Prop Let $R \hookrightarrow S$ be any inclusion of rings of char p. Suppose this splits. If S is Frobenius split then R is Frobenius-split.

Rk Splitting of the map $R^p \hookrightarrow R$ of R^p-modules is equivalent to the splitting of the map $R \hookrightarrow R^{1/p}$ of R-modules.

Proof (of the Prop.)

$$\begin{tikzcd} R^{1/p} \arrow[r, hook] \arrow[d, leftrightarrow] & S^{1/p} \arrow[d, bend left, "\pi \text{ S-linear } (\Rightarrow R\text{-linear})"] \\ R \arrow[r, hook] & S \arrow[u] \arrow[l, bend left, "\exists \varphi \text{ R linear}, \varphi(1)=1"] \end{tikzcd}$$

$\Rightarrow \varphi \circ \pi|_{R^{1/p}}$ is a splitting.

III F-regularity:

$$R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} R \longrightarrow \cdots$$

$$\cdots \hookrightarrow R^{1/p^2} \hookrightarrow R^{1/p} \hookrightarrow R \hookrightarrow R^p \hookrightarrow R^{p^2} \hookrightarrow R^{p^3} \hookrightarrow \cdots$$

WORLD of F-singularities: classifies singularities of R by analyzing the R-module structure of $\{R^{1/p^e}\}$

Thm (Kunz)
R is regular (coordinate ring of a smooth affine variety) $\iff R^{1/p}$ is (locally) free over R
$\iff R^{1/p^e}$ is locally free, $\forall e$

$$\boxed{R \text{ regular} \Rightarrow F\text{-regular} \Rightarrow F\text{-split}}$$

Def R is F-regular (in Hochster-Huneke terminology strongly F-regular) if $\forall$ nonzero $c \in R$ $\exists e$ s.t. $c^{1/p^e}R \subseteq R^{1/p^e}$ splits.
Equivalently: $\exists \varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ s.t. $\varphi(c^{1/p^e}) = 1$.
Rk: if $e=1$, we recover F-splitting

Ex/Prop: Every regular ring is F-regular.
Proof:
R local
By Nakayama's Lemma R^{1/p^e} is free over R
Need to find e s.t. c^{1/p^e} is part of a minimal generating set for R^{1/p^e} over R.
By Nakayama: $c^{1/p^e} \notin m R^{1/p^e} \iff c \notin m^{[p^e]}R$

Prop Let $R \hookrightarrow S$ be any inclusion of rings of char p. Suppose this inclusion is split. Then S is F-regular $\Rightarrow R$ is F-regular.

Thm R F-regular $\Rightarrow R$ CM, normal.

Cor: The Hochster-Roberts thm in char $p>0$.
Proof: $R = S^G \underset{\text{split}}{\hookrightarrow} k[x_1, \ldots, x_n]$
regular $\Rightarrow$ F-regular $\Rightarrow S^G$ is F-regular $\Rightarrow S^G$ is CM