MSRI

Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu **Email/Phone:** aseceleanu2@math.unl.edu

Speaker's Name: Irena Peeva

Talk Title: Infinite free resolutions

Date: 08/28/12 **Time:** 9:00 am / pm (circle one)

List 6-12 key words for the talk: Betti numbers, resolution, Koszul, complete intersection

Please summarize the lecture in 5 or fewer sentances: Much less is known about properties of infinite free resolutions than those of finite free resolution. The lectures discuss resolutions over ~~c~~, ~~[illegible]~~, complete intersections and Koszul rings.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Irena Peeva – Infinite free resolutions I

Notation: k = field (everything depends on characteristic, but this point shall not be emphasized)

$S = k[x_1, \dots x_n]$, $\deg x_i = 1$ $\forall i$ – standard grading

I = homogeneous ideal

$R = S/I$ graded ring; $R(-p)_i = R_{i-p}$ (R shifted p degrees). The generator 1 in $R(-p)$ is in deg p.

M = finitely generated graded R-module

1. Resolutions

Def A free resolution $\mathbb{F}$ of M over R is

$\mathbb{F}: \quad \dots\dots F_i \xrightarrow{d_i} F_{i-1} \to \dots \to F_1 \xrightarrow{d_1} F_0$ such that

1) $\mathbb{F}$ is an exact complex
2) $M = F_0/\operatorname{Im}(d_1)$

Existence: we will construct a free resolution of M inductively.

Step 0: Set $M_0 = M$. Choose a system of generators of M.

Set $F_0 = R^r$

Define $d_1 : F_0 = R^r \longrightarrow M$

$$\underset{\text{basis element}}{f_i} \longmapsto m_i$$

Step i. Assume we have F_{i-1} and d_{i-1}.

Set $M_i = \operatorname{Ker}(d_{i-1})$

Choose a system of generators $h_1 \dots h_s$ of M_i

Set $F_i = R^s$ and define $d_i : F_i \longrightarrow F_{i-1}$

$$\underset{\text{basis}}{g_j} \longmapsto h_j$$

Remark: this process may not terminate; then we get an infinite resolution.

Uniqueness: every two free resolutions of M are homotopic. We need to impose more conditions to have isomorphism (the condition needed is minimality).

Building a free resolution amounts to repeatedly solving systems of equations over R.
Observation: given $R^p \xrightarrow{A} R^q$, describing $\operatorname{Ker} A \Leftrightarrow$
$\Leftrightarrow$ solving the ~~[illegible]~~ system $A\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}=0$, where

$y_1, \dots, y_p$ take values in R

<u>Example</u>: $S=k[x,y]$, $J=(x^3, xy, y^5)$. Resolve S/J.
Answer:
$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x^2 & -y^4 \\ 0 & x \end{pmatrix}} S^3 \xrightarrow{(x^3 \ \ xy \ \ y^5) = d_1} S \qquad (1)$$
f_1, f_2, f_3 basis

<u>Step 1</u>:
Describe $\operatorname{Ker}(d_1) \Leftrightarrow d_1(y_1 f_1 + y_2 f_2 + y_3 f_3) = 0$
i.e. solve $y_1 x^3 + y_2 \cdot xy + y_3 y^5 = 0$
A computation shows that the generators of $\operatorname{Ker}(d_1)$ are $y f_1 - x^2 f_2$ and $-y^4 f_2 + x f_3$.

<u>Step 2</u>
Describe $\operatorname{Ker}(d_2) \Leftrightarrow d_2(Y_1 g_1 + Y_2 g_2) = 0$
i.e. solve
$$\begin{cases} Y_1 y = 0 \\ -x^2 Y_1 - y^4 Y_2 = 0 \\ x Y_2 = 0 \end{cases} \Rightarrow Y_1 = Y_2 = 0$$

A free resolution of M is a description of the structure of M:
$$\cdots \longrightarrow F_2 \xrightarrow{\left(\begin{smallmatrix}\text{a generating} \\ \text{system of} \\ \text{relations on} \\ \text{the relations} \\ \text{in } d_1\end{smallmatrix}\right)} F_1 \xrightarrow{d_1 = \left(\begin{smallmatrix}\text{a generating} \\ \text{system of} \\ \text{relations on the} \\ \text{generators of } M\end{smallmatrix}\right)} F_0 \xrightarrow{d_0 = \left(\begin{smallmatrix}\text{a system} \\ \text{of genera-} \\ \text{tors of } M\end{smallmatrix}\right)} M \longrightarrow 0$$

2. Minimality and Betti numbers

<u>Def</u> $\mathbb{F}$ is graded if the degree of the differential is 0
that is: F_i is graded $\forall i$ and if $f \in F_i$ is homo-

geneous ~~that~~ $\deg(d(f)) = \deg(f)$.
The graded version of resolution (1) is:

$$0 \to S(-4) \oplus S(-6) \longrightarrow S(-3) \oplus S(-2) \oplus S(-5) \to S \qquad (2)$$

$\mathbb{F}$ now has a bigrading F_{ij} has $\begin{cases} \text{homological degree } i \\ \text{internal degree } j \end{cases}$

$\mathbb{F}$ is exact $\iff \forall j \quad \dots \to F_{ij} \to F_{i-1,j} \longrightarrow \dots \to F_{0j}$ is an exact sequence of vector spaces

<u>Key point</u>: Nakayama's Lemma holds:
that is, if J is a proper graded ideal and $M = JM$, then $M = 0$

it implies:
<u>Foundational Theorem</u>:
Let $r = \dim \overline{M}$, where $\overline{M} = M/(x_1, \dots, x_n)M$ is a vector space.
Then every minimal system of homogeneous generators has r elements.
Homogeneous elements $m_1, \dots, m_r$ in M form a minimal system of generators $\iff \overline{m}_1, \dots, \overline{m}_r$ is a basis in $\overline{M}$.
We have this property in the local and graded cases.

<u>Def</u> A free resolution $\mathbb{F}$ of M is <u>minimal</u> if $d_{i+1}(F_{i+1}) \subseteq$
$\subseteq (x_1, \dots, x_n) F_i \quad \forall i$.
Minimal means — at every step in the construction we choose a <u>minimal</u> system of gens of $\operatorname{Ker}(d_{i-1})$ (in particular a graded free resolution exists)

<u>Thm</u> Let $\mathbb{F}$ be a minimal free resolution of M and $\mathbb{G}$ be a free res. of M. Then $\mathbb{G} \cong \mathbb{F} \oplus$ (short exact complex of the form: $0 \to R(-p) \xrightarrow{1} R(-p) \to 0$)
"consecutive cancellation"

The minimal free resolution is unique up to an iso.
<u>Def</u> $b_i^R(M) = \operatorname{rank} F_i$ in the minimal free res. of M ~~are~~

called the (total) Betti numbers of M.

<u>Def</u> The <u>projective dimension</u> $pd(M) = \max\{i \mid b_i^R(M) \neq 0\}$ is the length of $\mathbb{F}$.

<u>Thm</u> (Hilbert Syzygy Theorem)
$pd_S M$ ~~is~~ finite ($\leq n$).

Hence all minimal free resolutions over S are finite.

<u>Thm</u> (Auslander - Buchsbaum - Serre)
TFAE 1) $pd_R M < \infty$ for a given M
2) $pd_R M < \infty$ $\forall M$
3) R regular ($\Leftrightarrow I$ is generated by linear forms)

Over other rings, most minimal free resolutions are infinite.

<u>Example</u>: $R = k[x]/(x^2)$
$$\cdots \to R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \quad \text{resolves } k$$

How do infinitely many Betti numbers behave?

<u>Problem 1</u> (Avramov 1990)
What type of growth can $\{b_i^R(M)\}$ have?
Are polynomial and exponential growth the only possibilities?
- it is known that the growth is at most exponential
- it is known that ~~the~~ only polynomial growth occurs over complete intersections
- <u>Thm</u> (Gulliksen 1980) — Homological criterion for ci
  TFAE 1) $\{b_i^R(k)\}$ has polynomial growth
  2) $\{b_i^R(M)\}$ has polynomial growth $\forall M$
  3) R is a complete intersection

<u>Problem 2</u> (Avramov 1990)
Is it true that the Betti sequence of M is eventually non-decreasing? ($\forall M$)
- yes, over complete intersections (see Lecture III)

• yes for $\{ b_1^R(k) \}$ (Gulliksen 1980)