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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Bernhard Keller

Talk Title: Quiver Representations & Cluster Algebra

Date: 08/28/12 Time: 10:30 am / pm (circle one)

List 6-12 key words for the talk: quiver, representation, cluster algebra, Caldero-Chapoton map

Please summarize the lecture in 5 or fewer sentances: This lecture gives an introduction to representations of quivers and shows how they can be used to study cluster algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Bernhard Keller – Quiver representations and cluster algebras

Aim: solve the recurrence equations of cluster theory using representation theory
e.g. we would like to write down closed formulas for the cluster variables

Method of proof: categorification i.e. we lift the cluster combinatorics to the level of categories of representations.

Plan: 1. Quiver representations and cluster variables
2. Cluster categories
3. Quivers with potential (Derksen – Weyman – Zelevinsky)

1. Q a finite quiver e.g. $Q: 1 \xrightarrow{\alpha} 2$
Q_0 = the set of the vertices
Q_1 = the set of the arrows
k = an algebraically closed field

Def A representation V of Q is given by ~~a finite dim~~
- a finite dim k-vector space V_i, $i \in Q_0$
- a linear map $V_\alpha : V_i \to V_j$ $\forall$ arrow $\alpha : i \to j$ in Q

Ex A representation of $1 \to 2$ is a diagram
$V_1 \xrightarrow{V_\alpha} V_2$.

Def A morphism of representations $f: V \to W$ is given by linear maps $f_i : V_i \to W_i$, $\forall i \in Q_0$ such that for each arrow $\alpha : i \to j$ the square commutes

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"] & W_j \end{tikzcd}$$

Rk We obtain the category of representations $\mathrm{rep}_k(Q)$
It is an abelian category. In particular we have a notion of direct sum:

$(V\oplus W)_i = V_i \oplus W_i$, $(V\oplus W)_\alpha = V_\alpha \oplus W_\alpha : V_i \oplus W_i \to V_j \oplus W_j$

<u>Ex</u> Fix a representation $V = V_1 \xrightarrow{V_\alpha} V_2$ of $1 \to 2$

$$\begin{tikzcd}
V \arrow[d, "\text{iso }\wr"'] & V_1 \arrow[r, "V_\alpha"] & V_2 \\
V' & k^p \arrow[u, "\wr"] \arrow[r] & k^q \arrow[u, "\wr"']
\end{tikzcd}$$

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \qquad r = \mathrm{rk}(V_\alpha)$$

↳ canonical form

V' is isomorphic to
$(k \xrightarrow{1} k)^r \oplus (k \longrightarrow 0)^{p-r} \oplus (0 \to k)^{q-r}$

<u>Def</u> Let V be a representation. A <u>subrepresentation</u> $V' \subseteq V$ is given by a family of subspaces $V_i' \subseteq V_i$ $\forall i \in Q_0$ s.t. $V_\alpha(V_i') \subseteq V_j'$ $\forall$ $\alpha : i \to j$.

V is <u>simple</u> $\iff$ $V \neq 0$ and each subrepresentation equals 0 or V

V is <u>indecomposable</u> $\iff$ $V \neq 0$ and $\forall U, W$, $V \cong U \oplus W$ $\Rightarrow U = 0$ or $W = 0$.

<u>Thm</u> (Decomposition Theorem) Azumaya – Fitting – Krull Remak – Schmidt

a) V is indecomposable $\iff$ $\mathrm{End}(V)$ is local (i.e. its non-invertible elements form an ideal)

b) each representation decomposes into a direct sum of indecomposable representations and these are unique up to isomorphism and permutation.

<u>Example</u> By b), the complete list of indecomposable representations of $1 \to 2$ (up to isom.) is
$S_1 = (k \to 0)$, $P_1 = (k \xrightarrow{1} k)$, $S_2 = (0 \to k)$.

<u>Example</u> with infinitely many indecomposable representations

Q = Kronecker quiver $= (1 \rightrightarrows 2)$

Then $V(x) = \left(k \overset{x_0}{\underset{x_1}{\rightrightarrows}} k\right)$, $(x_0, x_1) \in \mathbb{P}^1(k)$ is an infinite family of pairwise non-isomorphic representations.

<u>Question</u>: which quivers have only finitely many indecomp./isom ?

From now: Q connected, $Q_0 = \{1, \dots, n\}$

<u>Thm</u> (Gabriel, 1972) Q has only finitely many indec./isom $\Longleftrightarrow$ Q ~~is~~ a Dynkin quiver i.e. Q is an orientation $\vec{\Delta}$ of a simply laced Dynkin diagram Δ.

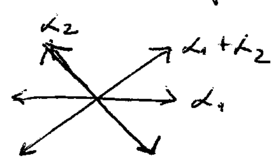
Moreover, if $Q = \vec{\Delta}$, then

$$\{\text{indec. repres. of } Q\}/\text{isom} \xrightarrow[\text{bij.}]{\sim} \{\text{positive roots of } \Delta\}$$

$$V \longmapsto \sum_{i=1}^{n} (\dim V_i)\,\alpha_i$$

where $\alpha_1, \dots, \alpha_n$ are the simple roots.

<u>Ex</u> $Q = (1 \longrightarrow 2)$ $\vec{A}_2$



$S_1 = (k \longrightarrow 0) \longleftrightarrow \alpha_1 \longleftrightarrow \dfrac{1 + x_2}{x_1}$

$P_1 = (k \xrightarrow{1} k) \longleftrightarrow \alpha_1 + \alpha_2 \longleftrightarrow \dfrac{x_1 + 1 + x_2}{x_1 x_2}$

$S_2 = (0 \longrightarrow k) \longleftrightarrow \alpha_2 \longleftrightarrow \dfrac{1 + x_1}{x_2}$

simple reps $\Downarrow$ 2 terms in the numerator

<u>Thm</u> (Fomin - Zelevinsky, 2003)

The cluster algebra $\mathcal{A}_Q$ associated to the quiver Q has finitely many cluster variables iff Q is a Dynkin quiver. Moreover, if $Q = \vec{\Delta}$, then

$$\{\text{positive roots of } \Delta\} \xrightarrow{\sim} \{\text{non initial cluster variables}\}$$

$$\alpha = \sum_{i=1}^{n} d_i \alpha_i \longmapsto \text{unique } x_\alpha = \frac{\cdots\cdots}{x_1^{d_1} \cdots x_n^{d_n}}$$

<u>Cor</u> If $Q = \vec{\Delta}$ then

$$\{\text{indec. reps}\}/\text{isom} \xrightarrow{\sim} \{\text{non initial cluster var.}\}$$

$$V \longmapsto \text{unique } x_V = \frac{\cdots}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $d_i = \dim V_i$, $\forall i$

Subreps of $k \xrightarrow{1} k$: $(0 \to 0), (0 \to k), (k \to k)$

__Obs__ If $Q = \vec{A_n}$, then # subreps of V = #(terms in the numerator of X_V)

This fails for $\vec{D_4}$.
We should count subrepresentations with suitable weights.

From now on $k = \mathbb{C}$.

__Def__ Let V be a representation of Q and $d = \dim V = (\dim V_i)_{i=1,\dots,n} \in \mathbb{N}^n$.
Let $0 \le e \le d$ (i.e. $0 \le e_i \le d_i$, $\forall i$)
$Gr_e(V)$ = quiver Grassmanian
$= \{ V' \le V \mid V' \text{ subreps such that } \dim V' = e \}$
closed
$\prod Gr_{e_i}(V_i)$

__Thm__ (Caldero – Chapoton, 2006)
Q Dynkin
V indecomposable representation

$$X_V = \underbrace{\frac{1}{x_1^{d_1} \cdots x_n^{d_n}} \cdot \sum_{0 \le e \le d} \chi(Gr_e(V)) \prod x_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)}}_{=: CC(V)}$$

where $\chi(Gr_e(V))$ = topological Euler characteristic

__Ex__: Q:
$$\begin{array}{ccc} 1 & \searrow & \\ 2 & \to & 4 \\ 3 & \nearrow & \end{array}$$

V
$$\begin{array}{ccc} \mathbb{C} & \searrow & \\ \mathbb{C} & \to & \mathbb{C}^2 \\ \mathbb{C} & \nearrow & \end{array}$$
3 lines in general position
$\Rightarrow V$ is indec of dim. vector $\begin{bmatrix} 1 & \\ 1 & 2 \\ 1 & \end{bmatrix}$

13 possible dim. vectors of subreps.
Among these 12 give $Gr_e(V) = pt$ and 1, namely $e = \begin{bmatrix} 0 & \\ 0 & 1 \\ 0 & \end{bmatrix}$
gives $Gr_e(V) = \mathbb{P}^1(\mathbb{C}) \Rightarrow \chi = 2$