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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Irena Peeva

Talk Title: Infinite free resolutions

Date: 08/29/12 Time: 9:00 am / pm (circle one)

List 6-12 key words for the talk: Betti numbers, resolution, Koszul, complete intersection

Please summarize the lecture in 5 or fewer sentances: Much less is known about properties of infinite free resolutions than those of finite free resolution. The lectures discuss resolutions over ~~c~~, ~~[illegible]~~, complete intersections and Koszul rings.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Irena Peeva - Infinite free resolutions II

Boundedness of $\{b_i^R(M)\}$ is a very strong condition.

Problem (Avramov) What rings admit bounded Betti numbers?

Problem: What can we say about the structure of $\mathbb{F}$ if $\{b_i^R(M)\}$ is bounded? (constant eventually).

(Over ci these rings must have period 2 resolutions, - Eisenbud but there are examples of rings with periodic resolutions of any period - Gasharov & Peeva)

Problem (Ramras 1980) If $\{b_i^R(M)\}$ is bounded, then is it true that it is eventually constant?

Subproblem: Does there exist a module M with periodic minimal free resolution and non-constant Betti numbers? ($\longrightarrow$ computer may help with this problem)

- (Eisenbud 1980) If the period is 2, then the Betti numbers are constant.

3. Rationality and Golod Rings

Poincaré series $P_M^R(t) = \sum_{i \geq 0} b_i^R(M)\, t^i$

Problem (Serre - Kaplansky) Is $P_k^R(t)$ rational?

Thm (Gulliksen 1972)

If $P_k^R(t)$ is rational for all R, then $P_M^R(t)$ is rational for all R and M.

Thm (Anick - Gulliksen 1985)

For any R, there exists a ring R' such that $(m')^3 = 0$ and $P_k^R(t)$ and $P_k^{R'}(t)$ are rationally expressible in terms of each other.

Anick constructed a counterexample (1982) to the Serre - Kaplansky ~~problem~~ problem. At the moment it is not well understood if "most" rings have rational or irrational Poincaré series.

Problem What happens generically?

- If I is generated by monomials $\Rightarrow P_k^R(t)$ is rational (Backelin 1982)
- If I is a toric ideal (an ideal J is <u>toric</u> if it is the kernel of $S \to k[m_1, \dots, m_n]$, $x_i \mapsto m_i$, where m_i is a monomial in $k[t_1, \dots, t_r]$)

- Roos - Sturmfels 1998: give a CM monomial curve in $\mathbb{P}^8$ where I is generated by quadrics and $P_K^R(t)$ is irrational.
- Gasharov - Peeva - Welker 2000: $P_k^R(t)$ is rational for generic toric ideals.

Idea: Relate the properties of $P_k^R(t)$ to the properties of $P_R^S(t)$.

Serre's Inequality: $P_k^R(t) \leq \dfrac{(1+t)^n}{1-t^2 P_I^S(t)}$

<u>Def</u> R is <u>Golod</u> if equality holds.

<u>Rk</u> Goldness is usually defined using Massey products on $H(\mathbb{K} \otimes S/I)$.

If R is Golod, then Golod constucted the minimal free resolution of k (expressed in terms of Massey products).

<u>Thm</u> 1) If I has a linear minimal free resolution over S $\Rightarrow R$ is Golod.

2) (Levin 1975) $(x_1, \dots, x_n)^p$ for $p \gg 0$ is Golod.

3) (Herzog - Welker - Yassemi 2012) I^p is Golod for $p \gg 0$.

4) Borel-fixed ideals are Golod (Peeva 1996, Aramova - Herzog 1996)

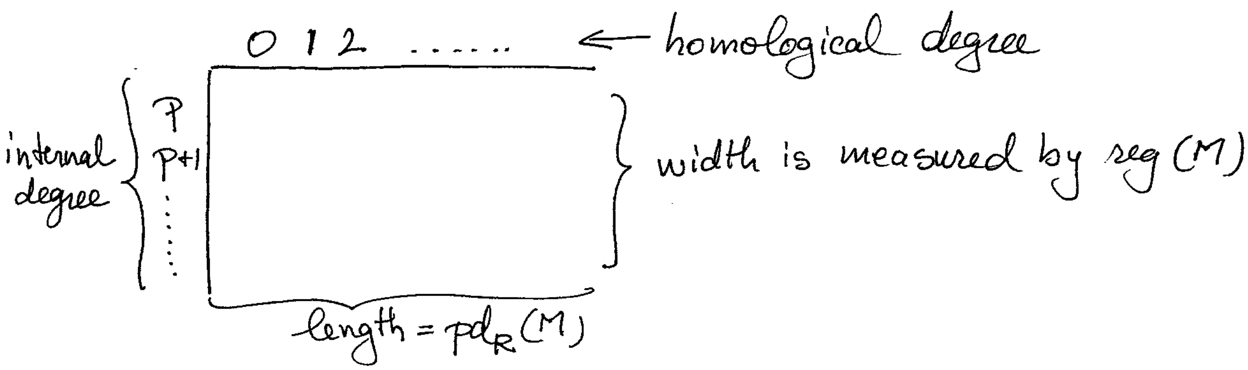
(A monomial ideal L is <u>Borel fixed</u> if $x_i m \in L \Rightarrow x_j m \in L \ \forall j \leq i$)

5) (Gasharov - Peeva - Welker 2000) generic toric ideals are Golod.

4. <u>Regularity and Rate</u>

Graded Betti numbers $b_{ij}^R(M) = \dim \operatorname{Tor}_i(M, k)_j$

The Betti table - the $(i,j)^{th}$ entry is $b_{i,i+j}$.



<u>Def</u> $reg_R(M) = \max\{ j \mid b^R_{i,i+j}(M) \neq 0 \}$

Regularity can be finite for an infinite minimal free res.

<u>Ex</u>

$$R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \quad \text{for } R = k[x]/(x^2)$$

$$\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \end{array}$$

is the Betti table

$$R \xrightarrow{x^2} R \xrightarrow{x^2} R \xrightarrow{x^2} R \quad \text{for } R = k[x]/(x^4)$$

$$\begin{array}{c|cccc} 1 & 1 & & & \\ 2 & & 1 & & \\ 3 & & & 1 & \\ \vdots & & & & \ddots \end{array}$$

below this line we have 0

has infinite regularity

(Backelin 1988) introduced

$rate_R(M) = \sup\left\{ \frac{p_i - 1}{i-1} \mid i \geq 2 \right\}$ where $p_i = \max\{ j \mid b^R_{ij}(M) \neq 0 \}$

(Aramova – Bărcănescu – Herzog 1995) rate $< \infty$

<u>Problem</u> Determine or find upper bounds for rate.

- if I is generated by monomials, then $rate_R(k) = (\text{max degree of a minimal generator}) - 1$ (Eisenbud – Reeves – Totaro 1994)
- if I is a generic toric ideal, the same holds.

When is $reg_R(M) < \infty$?

<u>Thm</u> ([Avramov – Eisenbud 1992] [Avramov – Peeva 2001])

TFAE
1) $reg_R(k) < \infty$
2) $reg_R(M) < \infty \ \forall M$
3) R is Koszul (def: R is Koszul if $reg_R(k) = 0$

i.e. the differentials in the minimal free resolution of k have linear entries).

(Ananyan – Hochster 2011) proved that the pd of r quadrics (r is fixed) over a polynomial ring (on any number of variables) is bounded.

<u>Problem</u>: Is regularity of ideals generated by r quadrics bounded over any Koszul ring?