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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Bernhard Keller

Talk Title: Quiver Representations & Cluster Algebra

Date: 08/28/12 Time: 10:30 am / pm (circle one)

List 6-12 key words for the talk: quiver, representation, cluster algebra, Caldero-Chapoton map

Please summarize the lecture in 5 or fewer sentances: This lecture gives an introduction to representations of quivers and shows how they can be used to study cluster algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Bernhard Keller – Quiver representations and cluster algebras

<u>Def</u> A quiver Q is <u>acyclic</u> if it does not have any oriented cycles (e.g.
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$
).

A representation V is <u>rigid</u> $\Leftrightarrow \mathrm{Ext}^1(V,V)=0$

<u>Ex</u> · Any indecomposable representation of a Dynkin quiver is rigid.
· The representations $\left(k \overset{x_0}{\underset{x_1}{\rightrightarrows}} k\right)$ $(x_0,x_1)\in\mathbb{P}^1(k)$ are not rigid.

<u>Thm</u> (Caldero – Keller, 2006) Q acyclic $\Rightarrow$

$$\{\text{indec. rigid reps}\}/\text{isom} \xrightarrow{\sim} \left\{\begin{matrix}\text{non-initial cluster}\\ \text{variables of } \mathcal{A}_Q\end{matrix}\right\}$$
$$V \longmapsto CC(V)$$

<u>Rk</u> 1) Reineke has shown (04/12) that each proj. variety is isomorphic to $Gr_e(V)$ for some acyclic quiver Q and some representation V.
2) If Q is acyclic and V is rigid, then $Gr_e(V)$ is smooth (Caldero – Reineke '06). Moreover in this case $\chi(Gr_e(V))\geq 0$ (Nakajima / Qin 2010). This confirms FZ's positivity conjecture (w.r.t. $x_1,\dots,x_n$).

<u>Questions</u> What about the initial variables $x_1,\dots,x_n$?
What about the clusters?
What about the exchange relations
$$u_k' u_k = \prod_{i\to k} u_i + \prod_{k\to j} u_j$$

Nice answers can be given using the cluster category.

2 <u>The cluster category</u>

2.1. <u>Quivers vs algebras</u>

$k=\bar{k}$ alg. closed, Q = finite quiver e.g.

$$\begin{tikzcd} & 3 \arrow[loop above, "\beta"] \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\gamma"] & & 2 \arrow[ll, "\sigma"] \end{tikzcd}$$

Def A *path* in Q = formal composition of arrows

e.g. $(2|\alpha|\beta|\delta|1)$

For each vertex i, we have the *lazy path*, $e_i = (i|i)$

Path algebra $\quad kQ = \bigoplus_{p \text{ path}} kp$

$$p \cdot q := \begin{cases} \text{concat}(p, q) & \text{if defined} \\ 0 & \text{else} \end{cases}$$

$$1 = \sum_{i \in Q_0} e_i$$

Example

Q	$\bullet$	$\circlearrowleft^{x}$	$x \circlearrowleft \bullet \circlearrowright y$	$\bullet \to \bullet \to \bullet \to \bullet$
kQ	k	$k[x]$	$k\langle x, y\rangle$	$\begin{bmatrix} * & & & 0 \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{bmatrix} \subseteq M_4(k)$

Rks 1) $\dim kQ < \infty \iff Q$ acyclic

2) $\operatorname{rep}_k(Q) \xrightarrow{\sim} kQ\text{-mod} = \{k\text{-finite dim. left } kQ\text{-modules}\}$

$$V \longmapsto \bigoplus_{i \in Q_0} V_i$$

Question Which are the finite dim algebras kQ? kQ/I?

Prop (Gabriel) Let A be a finite dim alg $/ k = \bar{k}$.
Then there is a unique quiver Q_A s.t.
$$A\text{-mod} \xrightarrow{\sim} (kQ_A/I)\text{-mod}$$
for some ideal $I < kQ$ which is contained in the square of the ideal generated by the arrows of Q_A.
Moreover $I = 0 \iff A$ hereditary (i.e. $\operatorname{Ext}^2_A(-) = 0$)

Def / Prop Q_A = quiver of A

vertices $i \longleftrightarrow$ simple A-modules S_i

$\#(i \to j) = \dim \operatorname{Ext}^1(S_i, S_j)$

Example $A = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right] \Big/ \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \implies Q_A \quad 1 \rightrightarrows \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ $; I = 0$

2.2. <u>The cluster category</u>

Q a finite acyclic quiver, $k = \bar{k}$

<u>Aim</u>: construct a category $\mathcal{C}_Q$ = cluster category whose indecomposable rigid objects are in bijection with all cluster variables of $\mathcal{A}_Q$.

$\mathrm{mod}(kQ) = \{k\text{-fin. dim } kQ\text{-right modules}\}$

$\mathcal{D}_Q$ = bounded derived category $\mathrm{mod}(kQ)$

objects: bounded complexes of $M^p \in \mathrm{mod}\, kQ$

$$\to 0 \to \dots \to M^p \to M^{p+1} \to \dots \to 0 \to$$

morphisms: obtained from morph. of complexes by formally inverting all quasi-isomorphisms

<u>Rks</u> 1) We have a full embedding

$$\mathrm{mod}\, kQ \hookrightarrow \mathcal{D}_Q$$

$$M \longmapsto (\dots 0 \to \underset{\deg 0}{M} \to 0 \quad)$$

$M \in \mathrm{mod}\, kQ \Rightarrow \mathrm{Ext}^p_{kQ}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_Q}(L, \Sigma^p M)$, $p \in \mathbb{Z}$

where $(\Sigma X)^p = X^{p+1}$, $d_{\Sigma X} = -d_X$

2) $\mathcal{D}_Q$ is abelian $\Leftrightarrow$ Q has no arrows

But $\mathcal{D}_Q$ is always triangulated i.e. additive and endowed with

a) a suspension functor, namely $\Sigma: \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$

b) triangles $X \to Y \to Z \to \Sigma X$ namely those induced by short exact sequences of complexes satisfying certain axioms.

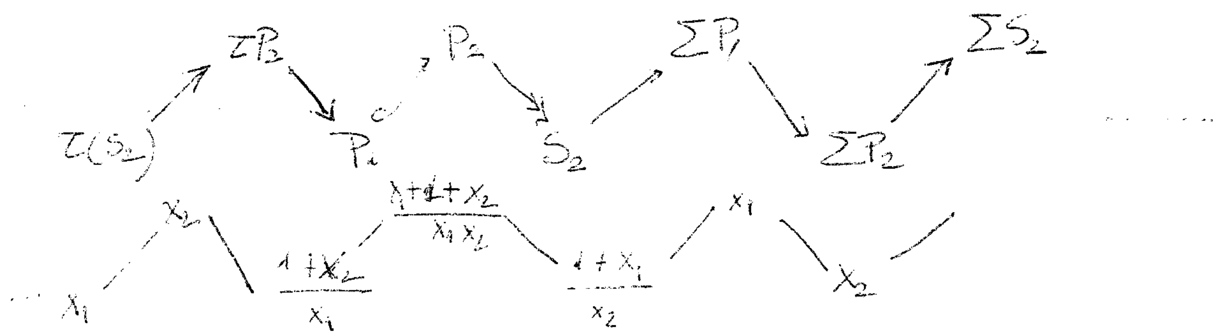
3) The decomposition thm holds for $\mathcal{D}_Q$ and the indecomposable objects are exactly the $\Sigma^p M$, M indec. module, $p \in \mathbb{Z}$.

<u>Example</u> $Q = (1 \to 2)$,

indec. modules $\quad S_1 = P_1 = e_1 kQ$

$P_2 = e_2 kQ$,

$S_2 = P_2 / P_1$



<u>Aim</u> characterize τ (= shift by 1 unit) intrinsically.

<u>Lemma</u>: $\mathcal{D}_Q$ has a Serre functor i.e. an auto-equivalence $S: \mathcal{D}_Q \to \mathcal{D}_Q$ s.t. $D\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(Y, SX)$ $\forall X, Y$

$D = \mathrm{Hom}_k(-, k)$

<u>Def</u> τ = Auslander-Reiten translation

$:= \Sigma^{-1} S : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$

$\mathcal{C}_Q$ = cluster category

= orbit category $\mathcal{D}_Q/(\tau^{-1}\Sigma)^{\mathbb{Z}} = \mathcal{D}_Q/(\Sigma^2 S^{-1})$

objects: same as $\mathcal{D}_Q$

morph: $\mathrm{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_Q}(X, (\tau^{-1}\Sigma)^p Y)$

<u>Rk</u> $\mathcal{C}_Q$ is due to (independently) — Caldero-Chapoton-Schiffler (for $Q = \vec{A}_n$)
— Buan-Marsh-Reiten-Reineke-Todorov

<u>Prop</u>. a) $\mathcal{C}_Q$ has fin. dim. morphism space and the decomposition thm. holds

b) $\mathcal{C}_Q$ is canonically triangulated and has a Serre functor $S_{\mathcal{C}_Q}$ induced by that of $\mathcal{D}_Q$ and therefore $S_{\mathcal{C}_Q} \cong \Sigma^2_{\mathcal{C}_Q}$ i.e. $\mathcal{C}_Q$ is 2-Calabi-Yau.

Consequence $\boxed{\mathrm{Ext}^1_{\mathcal{C}_Q}(X,Y) = \mathrm{Hom}_{\mathcal{C}_Q}(X, \Sigma Y) \cong D\,\mathrm{Ext}^1(Y,X)}$