MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecele anu2@math.unl.edu

Speaker's Name: Bernhard Keller

Talk Title: Quiver Representations & Cluster Algebra

Date: 08/30/12 Time: 09:00 am / pm (circle one)

List 6-12 key words for the talk: quiver, representation, cluster algebra, Caldero-Chapoton map

Please summarize the lecture in 5 or fewer sentances: This lecture gives an introduction to representations of quivers and shows how they can be used to study cluster algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Bernhard Keller – Quiver representations & cluster algebras III

Correction to section 2.1

Example

$$A = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \Rightarrow \begin{cases} Q_A : \; 2 \leftarrow 1 \rightarrow 3 \\ I = 0 \\ \text{so } A \text{ is hereditary} \end{cases}$$

Recall $k = \mathbb{C}$, Q = acyclic quiver on n vertices

kQ = path algebra

$\operatorname{mod}(kQ) = \{k\text{-fin. dim. right } kQ\text{-modules}\}$

$\downarrow$

$\mathcal{D}_Q$ = bounded derived category $= \mathcal{D}_Q/(S^{-1}\Sigma)^{\mathbb{Z}}$

$Q : 1 \to 2$

$$\begin{tikzcd}[column sep=small]
\operatorname{ind}(\operatorname{mod} kQ) \arrow[dd] & & & P_2 \arrow[dr] & \\
& & P_1 \arrow[ur] & & S_2 \\
\operatorname{ind}(\mathcal{D}_Q) \arrow[dd] & \cdots \arrow[dr] & & \Sigma P_1 \arrow[dr] & & P_2 \arrow[dr] & & \Sigma P_1 \arrow[dr] & & \Sigma S_2 \\
& & \Sigma P_2 \arrow[ur] & & P_1 \arrow[ur] & & S_2 \arrow[ur] & & \Sigma P_2 \arrow[ur] & \\
\operatorname{ind}(\mathcal{C}_Q) \arrow[dd, "\text{CC}"', "\wr"] & & P_1 \arrow[r] & P_2 \arrow[r] & S_2 \arrow[d] \\
& & \Sigma P_2 \arrow[u] & & \Sigma P_1 \arrow[ll] \\
\text{cluster variables} & & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& \frac{x_1+1+x_2}{x_1 x_2} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1+x_2}{x_1} \arrow[d, dash] & & \frac{1+x_1}{x_2} \arrow[d, dash] \\
x_2 \arrow[rr, dash] & & x_1
\end{tikzcd}$$

Example of a cluster category "in nature"

$G = \mathbb{Z}/3\mathbb{Z}$ acting on $S = \mathbb{C}[x,y,z]$ by $\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$,

ω = 3rd primitive root of 1
Let $R = S^G$
$\underline{CM}(R)$ = stable cat. of max CM-modules $\xrightarrow[\text{triangle}]{\sim} \mathcal{C}_Q$

where $Q: 1 \equiv 2$ (Keller - Reiten, 2008)
Generalization: Amiot - Iyama - Reiten (04/11)
Thanhoffer de Völcsey - Van den Bergh (06/11)

Back to cluster algebras

$$\begin{tikzcd} & \operatorname{mod}(kQ) \arrow[dl] \arrow[d] \arrow[dr, "CC"] & \\ \mathcal{D}_Q \arrow[r] & \mathcal{C}_Q \arrow[r, dashed] & \mathcal{A}_Q \end{tikzcd}$$

$CC(u \oplus v) = CC(u) \cdot CC(v)$
$\mathcal{A}_Q$ = cluster algebra
CC = natural extension

<u>Thm</u> a) $\left\{ \begin{matrix} \text{indec. rigid} \\ \text{objects of } \mathcal{C}_Q \end{matrix} \right\} / \text{isom} \xrightarrow[CC]{\sim} \left\{ \begin{matrix} \text{all cluster} \\ \text{variables of } \mathcal{A}_Q \end{matrix} \right\}$

$\Sigma P_i \longmapsto x_i$

$\{\text{rigid objects}\}/\text{isom} \xrightarrow{\sim} \left\{ \begin{matrix} \text{cluster} \\ \text{monomials} \end{matrix} \right\}$

b) Let $T_1, \dots, T_n$ be <u>pairwise</u> nonisomorphic indec. rigid objects in $\mathcal{C}_Q$. Then
$u_1 = CC(T_1), \dots, u_n = CC(T_n)$ form a cluster
iff $T = \bigoplus_{i=1}^{n} T_i$ is a cluster-tilting object, i.e.
T is rigid and has n pairwise non isomorphic indecomposable summands.

In this case, in <u>the</u> seed $(R, \{u_1, \dots, u_n\})$
(cf Gekhtman - Shapiro - Vainshtein) we have $R \cong Q_{End(T)}$

c) Two indecomposable rigid objects T_K, T_K' correspond to an exchange pair $u_K = CC(T_K)$, $u_{K'} = CC(T_K')$
iff $\dim Ext^1(T_K, T_K') = 1$.

In this case, in the exchange relation

$u_K \cdot u_K' = m + m'$
we have $m = CC(E)$ and $m' = CC(E')$
where $T_K \to E \to T_K' \xrightarrow{\neq 0} \Sigma T_K$ are non split
$T_K' \to E' \to T_K \xrightarrow{\neq 0} \Sigma T_K$ triangles

Rk: Surprisingly, most of the Thm generalizes to arbitrary quivers (w/o loops nor 2-cycles).
(Derksen – Weyman – Zelevinsky 2008, 2010)
Essential new ingredient: quivers w/ potential.

3. Quivers with potential

Q = finite quiver w/o loops nor 2-cycles

OK.

Completed path algebra $\widehat{\mathbb{C}Q} = \prod_{p \text{ path}} \mathbb{C}p$

Hochschild homology: $HH_0(\widehat{\mathbb{C}Q}) = \widehat{\mathbb{C}Q} \Big/ \left(\begin{smallmatrix}\text{closure of}\\ [\widehat{\mathbb{C}Q}, \widehat{\mathbb{C}Q}]\end{smallmatrix}\right)$

$\simeq \prod_{c \text{ cycles}} \mathbb{C}c$

Potential on Q: element $W \in HH_0(\widehat{\mathbb{C}Q})$ not involving cycles of length ≤ 2

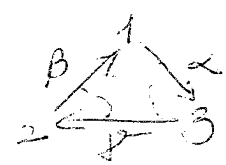
Example $W = \alpha\beta\gamma$

Cyclic derivatives: for each arrow $\alpha \in Q$, we have
$\partial_\alpha : HH_0(\widehat{\mathbb{C}Q}) \to \widehat{\mathbb{C}Q}$, $p \mapsto \sum_{p = u\alpha v} vu$

Eg $\partial_\beta(\alpha\beta\gamma) = \gamma\alpha$

Jacobian algebra: $J_{Q,W} = \widehat{\mathbb{C}Q} \Big/ \left(\begin{smallmatrix}\text{closed ideal gen.}\\ \text{by all } \partial_\alpha(W),\ \alpha \in Q\end{smallmatrix}\right)$

Eg $J_{Q,W} = \widehat{\mathbb{C}Q} / (\alpha\beta, \beta\gamma, \gamma\alpha)$



A right equivalence $(Q,W) \to (Q',W')$ is an alg. isom. $\varphi: \widehat{\mathbb{C}Q} \to \widehat{\mathbb{C}Q'}$ s.t. φ is the identity on the lazy paths.

Thm (DWZ) The mutation operation admits a

good extension to quivers with potential i.e. if
$\mu_j(Q,W) = (Q', W')$ (up to right eq.)
then $Q' = \mu_j(Q)$ at least for <u>generic</u> potential.

<u>Example</u> $\left(\begin{tikzcd}[column sep=small] & 1 \arrow[dr, "\alpha"] & \\ 2 \arrow[ur, "\beta"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}, W = \alpha\beta\gamma \right) \overset{\mu_1}{\longleftrightarrow} \left(\begin{tikzcd}[column sep=small] & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd}, W' = 0 \right)$

4. <u>Mutation of representations</u>

<u>Example</u> $Q: \begin{tikzcd}[row sep=small] & 2 \\ 1 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd} \xrightarrow[\mu_1]{} Q': \begin{tikzcd}[row sep=small] 2 \arrow[dr] & \\ & 1' \\ 3 \arrow[ur] & \end{tikzcd}$

Bernstein - Gelfand - Ponomarev reflection functors

$$\begin{tikzcd}[row sep=small] & V_2 \\ V_1 \arrow[ur] \arrow[dr] & \\ & V_3 \end{tikzcd} \qquad \begin{tikzcd} kQ\text{-mod} \arrow[d, shift right, "F"'] \\ kQ'\text{ mod} \arrow[u, shift right, "G"'] \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & V_2 \\ V_2 \prod_{V_1'} V_3 \arrow[ur] \arrow[dr] & \\ & V_3 \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] V_2 \arrow[dr] & \\ & V_2 \oplus_{V_1} V_3 \\ V_3 \arrow[ur] & \end{tikzcd} \qquad\qquad \begin{tikzcd}[row sep=small] V_2 \arrow[dr] & \\ & V_1' \\ V_3 \arrow[ur] & \end{tikzcd}$$

<u>Notice</u> F, G are not equivalences but induce equivalences $\mathcal{D}_Q \underset{RG}{\overset{LF}{\rightleftarrows}} \mathcal{D}_{Q'}$ (Happel '86)

General case
$$\mu_j(Q,W) = (Q', W')$$
We would like
$$\mathcal{D}(J_{Q,W}\text{-mod}) \xrightarrow{\sim} \mathcal{D}(J_{Q',W'}\text{-mod})$$

Unfortunately this is wrong already in our basic example. This can be repaired by considering the Ginzburg DG-algebra associated with (Q,W):
Graded quiver $\tilde{Q}$

$\deg \alpha = 0 = \dots$
$\deg \alpha^* = -1, \dots$
$\deg t_i = -2$

$$\begin{tikzcd} & 1 \arrow[loop above, "t_1"] \arrow[dl, shift right, "\beta^*"'] \arrow[dr, shift left, "\alpha"] & \\ 2 \arrow[loop left, "t_2"] \arrow[ur, shift right, "\beta"'] \arrow[rr, shift right, "\gamma^*"'] & & 3 \arrow[loop right, "t_3"] \arrow[ul, shift left, "\alpha^*"] \arrow[ll, shift right, "\gamma"'] \end{tikzcd}$$

$$\Gamma_{Q,W} = \left(\begin{array}{l} \text{completed graded path alg of } \tilde{Q} \text{ endowed with } d \text{ s.t.} \\ d(t_i) = e_i \left(\sum_{\alpha \in Q} [\alpha, \alpha^*] \right) e_i \\ d(\alpha^*) = \partial_\alpha (W) \end{array} \right)$$

Rk $H^0(\Gamma_{Q,W}) = J_{Q,W}$ but $\Gamma_{Q,W}$ has better homological properties

Thm (Keller - Yang, 2010)
If $\mu_j(Q,W) = (Q',W')$ then we have two equivalences

$$\mathcal{D}(\Gamma_{Q,W}) \underset{\phi^-}{\overset{\phi^+}{\rightrightarrows}} \mathcal{D}(\Gamma_{Q',W'})$$

("pretend that j is a source or a sink")
Fortunately ϕ^+ and ϕ^- induce the same equivalence in the generalized cluster category.

$\mathcal{C}_{Q,W} = \mathrm{per}(\Gamma_{Q,W}) / \mathcal{D}_{fd}(\Gamma_Q)$ where

$\mathrm{per}\,\Gamma_{Q,W}$ = closure of the free module $\Gamma_{Q,W}$ under shifts, extensions, direct factors in $\mathcal{D}\Gamma_{Q,W}$

$\mathcal{D}_{fd}\,\Gamma_{Q,W} = \{ X \in \mathcal{D}\Gamma_{Q,W} \mid \sum_{p \in \mathbb{Z}} \dim H^p X < \infty \}$

Concrete, almost equivalent description:
Derksen - Weyman - Zelevinsky: decorated representations.