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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Bernard Leclerc

Talk Title: Preprojective algebras and Lie theory

Date: 08/30/12 Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: preprojective algebra, Lie theory

Please summarize the lecture in 5 or fewer sentances: Several cluster algebras appear in Lie theory (e.g. Grassmanians). They can be understood by relating them to certain categories of modules over a preprojective algebra.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Bernard Leclerc – Preprojective algebras and Lie theory I

References. Geiss–Leclerc–Schröer: 0804.3168
1208.5749

G = a simple algebraic group / $\mathbb{C}$ of type A, D, E
$N \subset G$ = a maximal unipotent subgroup

Berenstein – Fomin – Zelevinsky: $\mathbb{C}[N]$ has a cluster algebra structure

$$\mathbb{C}[N] \xleftrightarrow{\text{Hopf dual}} U(n) \quad , n = Lie(N)$$

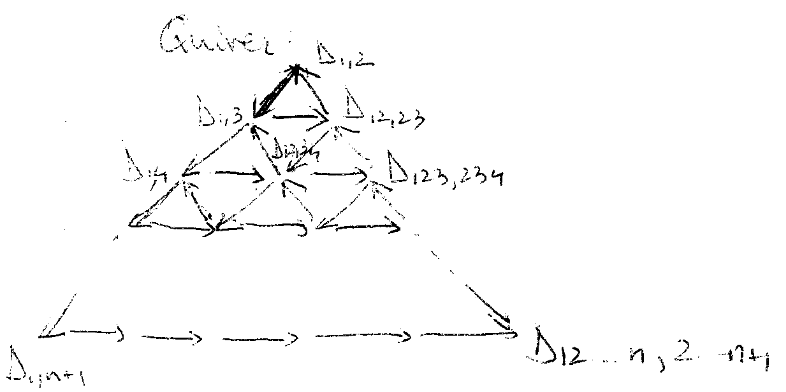
$\mathbb{C}[N] \supset$ Lusztig's dual canonical basis; $U(n) \supset$ Lusztig's canonical basis

Problem: Describe the cluster algebra structure and its connection with Lusztig's basis.

1. $\mathbb{C}[N]$ in type A_n

N = {upper unitriangular $(n+1)\times(n+1)$ matrices}
$\mathbb{C}[N] = \mathbb{C}[x_{ij} \mid 1 \le i, j \le n+1]$
If $I, J \subset \{1, \dots, n+1\}$ with same cardinality
$\Delta_{I,J}$ = minor of $\begin{bmatrix} 1 & x_{12} & \cdots & x_{1,n+1} \\ & \ddots & & \vdots \\ & & & x_{n,n+1} \\ 0 & & & 1 \end{bmatrix}$ taken on rows I, columns J

Quiver:



$\leftarrow$ all variables on the last row are frozen (coefficients)

Thm [BFZ] This is an initial seed for a cluster structure on $\mathbb{C}[N]$ (type A_n).

Example: $SL(3)$ A_2 $N = \begin{bmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{bmatrix}$

$$\begin{tikzcd} & x_{12} \arrow[dl] & \\ x_{13} \arrow[rr] & & x_{12}x_{23} - x_{13} = \Delta_{12,23} \arrow[ul, dashed] \end{tikzcd}$$

One mutation only:

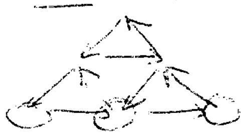
$x_{12}x_{23} = x_{13} + (x_{12}x_{23} - x_{13})$

2 mutable cluster variables x_{12}, x_{23}

$\{x_{12}^a x_{13}^b \Delta_{12,23}^c \mid a,b,c \in \mathbb{N}\} \cup \{x_{23}^a x_{13}^b \Delta_{12,23}^c \mid a,b,c \in \mathbb{N}\}$

= Lusztig's dual canonical basis!

Ex A_3 $SL(4)$



cluster algebra of type A_3 (acyclic)

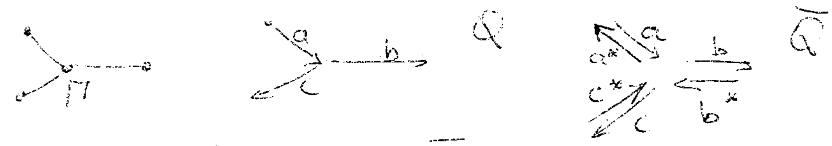
9 cluster variables $= \left\{ \begin{matrix} \Delta_{1,2}, \Delta_{1,3}, \Delta_{12,23}, \Delta_{4,3}, \Delta_{4,4} \\ \Delta_{2,4}, \Delta_{23,34}, \Delta_{12,24}, \Delta_{13,34} \end{matrix} \right\}$

Lie type of G	cluster type of $\mathbb{C}[N]$
A_2	A_1
A_3	A_3
A_4	D_6
all other types	infinite

2. Preprojective algebra

$G \rightsquigarrow \Gamma$ Dynkin diagram $\rightsquigarrow \overrightarrow{\Gamma}^{\text{orient}} = Q \rightsquigarrow \overline{Q}$ = double quiver

e.g. D_4



$\mathbb{C}\overline{Q}$ = path algebra

$\rho := \sum_{a \in Q_1} (aa^* - a^*a) \in \mathbb{C}\overline{Q}$

$\langle \rho \rangle$ = double sided ideal generated by ρ

Def (Gelfand – Ponomarev '79) $\Lambda := \mathbb{C}\overline{Q} / \langle \rho \rangle$

Properties 1) Regular repres of Λ considered as a repr. of Q:

$$\Lambda/\mathbb{C}G \cong \bigoplus_{\substack{X \text{ indec} \\ \text{of } G}} X \xRightarrow{\text{Gabriel's Thm}} \Lambda \text{ is finite dim.}$$

2) Λ is a self-injective algebra (injectives = projectives)
3) For any $X, Y \in \operatorname{mod}\Lambda$ $\operatorname{Ext}^1_\Lambda(X,Y) \cong D\operatorname{Ext}^1(Y,X)$
($\underline{\operatorname{mod}}\,\Lambda$ is 2 Calabi-Yau)
4)

	# cluster var.	# indec. rep of Λ
A_2	$2 + 2$ ← frozen	$2 + 2$ ← proj / inj
A_3	$9 + 3$	$9 + 3$
A_4	$36 + 4$	$36 + 4$
all other	∞	∞

3. $\underline{\text{A map } \operatorname{mod}\Lambda \rightsquigarrow \mathbb{C}[N]}$

$\underline{\text{Notation}}$ $I = \{\text{vertices of } \bar{Q}\} \xrightarrow{\sim} \{S_i \text{ simple } \Lambda\text{-modules}\}$

$\underline{d} = (d_i)_{i\in I} = \mathbb{N}^I$ (dim. vector for Λ)

$X \in \operatorname{mod}\Lambda$ with dimension vector $\underline{d}$

$d = \sum_{i\in I} d_i = \dim X$, $\underline{i} = (i_1, \dots, i_d) \in I^d$

$\mathcal{F}_{X,\underline{i}} = \{ f = (0 = F_0 \subset F_1 \subset \dots \subset F_d = X)$ the F_i's are sub Λ-modules
$F_k / F_{k-1} \cong S_{i_k} \}$
$= \{$composition series of X of type $\underline{i}\} \subset_{\text{closed}} \mathrm{Flag}(X)$

$\mathcal{F}_{X,i}$ is a complex projective variety

$x_i(t) = \exp(t e_i)$, $t \in \mathbb{C}$, $i \in I$, e_i Chevalley generator of $n = \mathrm{Lie}(N)$

one-parameter subgroups of N

They generate the group N.

To specify $f \in \mathbb{C}[N]$ it is enough to express $f(x_{i_1}(t_1), \dots, x_{i_k}(t_k))$ for any $\underline{i} = (i_1, \dots, i_k)$
$\underline{t} = (t_1, \dots, t_k)$

$\underline{\text{Thm}} / \underline{\text{Def}}$ Let X be a finite dim Λ-module

There exists a unique $f_X \in \mathbb{C}[N]$ s.t.

$$f_X(x_{\underline{i}}(\underline{t})) = \sum_{\underline{a}\in\mathbb{N}^k} \chi(\mathcal{F}_{X,\underline{i}^{\underline{a}}}) \frac{\underline{t}^{\underline{a}}}{\underline{a}!}$$

$\underline{i}^{\underline{a}} = (\underbrace{i_1 \dots i_1}_{a_1}, \underbrace{i_2 \dots i_2}_{a_2}, \dots)$, $\frac{\underline{t}^{\underline{a}}}{\underline{a}!} = \frac{t_1^{a_1} \dots t_k^{a_k}}{a_1! \dots a_k!}$