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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Bernard Leclerc

Talk Title: Preprojective algebras and Lie theory #2

Date: 08/30/12 Time: 09:00 am / pm (circle one)

List 6-12 key words for the talk: preprojective algebra, Lie theory

Please summarize the lecture in 5 or fewer sentences: Several cluster algebras appear in Lie theory (e.g. Grassmanians). They can be understood by relating them to certain categories of modules over a preprojective algebra.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Bernard Leclerc – Preprojective algebras and Lie theory II

Example A_2 $\Lambda: 1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2$ relations $0 = aa^*$, $a^*a = 0$

S_1 $\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0$

S_2 $0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}$

P_1 $\mathbb{C} \xrightarrow{id} \mathbb{C}$

P_2: $\mathbb{C} \xleftarrow{id} \mathbb{C}$

These are the only indecomposable Λ-modules.

$\underline{i} = (1,2,1)$

$$x_{\underline{i}}(\underline{t}) = \underbrace{\begin{bmatrix} 1 & t_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{x_1(t_1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ & 1 & t_2 \\ & & 1 \end{bmatrix}}_{x_2(t_2)} \underbrace{\begin{bmatrix} 1 & t_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{x_1(t_3)} = \begin{bmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\varphi_{S_1}(x_i(t)) = t_1 + 0 + t_3$ $a \in \{(1,0,0), (0,1,0), (0,0,1)\}$

$\varphi_{S_2}(x_i(t)) = t_2$

$\Longrightarrow \varphi_{S_1} = \Delta_{1,2}$

$\varphi_{S_2} = \Delta_{2,3}$

$\varphi_{P_1}(x_{\underline{i}}(\underline{t})) = t_2 t_3$ $\varphi_{P_1} = \Delta_{12,23}$

$\underline{a} = (1,1,0)$ or $\underline{a} = (0,1,1)$

| $\varphi_{P_2}(x_{\underline{i}}(t)) = t_1 t_2$

$\varphi_{P_2} = \Delta_{1,3}$

$\varphi_{S_1 \oplus P_2}(x_{\underline{i}}(t)) = 2t_1^2 t_2 / (2!\ 1!) + t_1 t_2 t_3 = t_1 t_2 (t_1 + t_3)$

$\underline{a} = (2,1,0)$ $\mathcal{F}_{S_1 \oplus P_2, \underline{i}\underline{a}} \cong \mathbb{P}^1$

$\underline{a} = (1,1,1)$ —— " —— $=$ point

$\underline{a} = (0,1,2)$ —— " —— $=$ empty

$\Longrightarrow \varphi_{S_1 \oplus P_2} = \varphi_{P_2} \cdot \varphi_{S_1}$

Proof of the existence of φ_X (sketch)

Lusztig's Lagrangian construction of $U(\mathfrak{n})$ (1991)

Fix $\underline{d} \in \mathbb{N}^I$

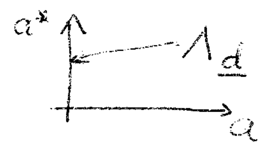
$\Lambda_{\underline{d}}$ = affine variety of representations of Λ with dimension vector $\underline{d}$.

$\subseteq_{\text{closed}} \operatorname{rep}(\overline{Q}, \underline{d})$

finite representations

(Ex A_2 relations $aa^* = a^*a = 0$

$d = (1,1)$


)

For $\underline{i} \in I^d$ $(d = \sum d_i)$

$$\chi_{\underline{i}} : \Lambda_{\underline{d}} \longrightarrow \mathbb{Z}, \qquad X \longmapsto \chi(\mathcal{F}_{X,\underline{i}})$$

$\chi_{\underline{i}}$ is constructible i.e. it takes finitely many values and the fibers are constructible subsets

$\mathcal{M}_{\underline{d}}$ = $\mathbb{C}$-subspace of ~~all~~ the vector space of all $\mathbb{C}$-valued constructible functions on $\Lambda_{\underline{d}}$ spanned by the $\chi_{\underline{i}}$.

$\mathcal{M} := \bigoplus_{\underline{d} \in \mathbb{N}^I} \mathcal{M}_{\underline{d}}$, a $\mathbb{C}$-vector space

Make $\mathcal{M}$ into an algebra by defining a convolution product: $f' \in \mathcal{M}_{\underline{d}'}$, $f'' \in \mathcal{M}_{\underline{d}''}$

$$\underbrace{(f' * f'')}_{\mathcal{M}_{\underline{d}'+\underline{d}''}}\big(\underbrace{X}_{\Lambda_{\underline{d}'+\underline{d}''}}\big) := \int_{Y \text{ submodule of } X} f'(Y)\, f''(X/Y)\, d\chi \overset{\text{def}}{=} \sum_{m \in \mathbb{C}} m\, \chi\left(\left\{ Y \subset X \;\middle|\; \begin{array}{l} f'(Y) \\ f''(X/Y) = m \end{array},\ \underline{\dim}\, Y = \underline{d}' \right\}\right)$$

<u>Thm</u> (Lusztig)

(i) $(\mathcal{M}, *)$ is an associative algebra

(ii) The assignment $e_{i_1} \cdots e_{i_d} \longmapsto \chi_{\underline{i}}$ extends to algebra isomorphism $U(\mathfrak{n}) \xrightarrow{\sim} (\mathcal{M}, *)$

For $X \in \operatorname{mod} \Lambda$, $\underline{\dim}\, X = \underline{d}$ we have a linear form

$$\delta_X : \mathcal{M}_{\underline{d}} \longrightarrow \mathbb{C} \qquad \in \mathcal{M}_{\underline{d}}^*, \qquad f \longmapsto f(X)$$

$$\mathcal{M}^*_{\text{graded}} = \bigoplus \mathcal{M}_{\underline{d}}^* \xrightarrow{\sim} \mathbb{C}[N], \qquad \delta_X \longmapsto \varphi_X$$

4. Properties of φ

Multiplicative properties of φ come from properties of the comultiplication of $\mathcal{M}$. These have not been studied by Lusztig.

<u>Thm</u> [Geiss - Leclerc - Schröer]

(i) $\varphi_{X \oplus Y} = \varphi_X \cdot \varphi_Y$

(ii) If $\dim(\mathrm{Ext}^1_\Lambda(X,Y)) = 1$

$0 \to X \to E \to Y \to 0$ } nonsplit short exact seq.

$0 \to Y \to E' \to X \to 0$ } (E, E' are unique up to iso)

$\Rightarrow \varphi_X \varphi_Y = \varphi_E + \varphi_{E'}$

<u>Example</u>: type A_2

$\dim(\mathrm{Ext}^1(S_1, S_2)) = 1$

$S_1 \hookrightarrow P_2 \twoheadrightarrow S_2$

$S_2 \hookrightarrow P_1 \twoheadrightarrow S_1$

$\rightsquigarrow \varphi_{S_1} \varphi_{S_2} = \varphi_{P_1} + \varphi_{P_2}$.