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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asceleanu2@math.unl.edu

Speaker's Name: Frank-Olaf Schreyer

Talk Title: Syzygies, finite length modules and random curves #1

Date: 08/31/12 Time: 2:00 am / pm (circle one)

List 6-12 key words for the talk: Gröbner basis, Hilbert syzygy theorem, Petri's theorem.

Please summarize the lecture in 5 or fewer sentances: Gröbner basis theory is applied to proving Hilbert's syzygy theorem and Petri's theorem.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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Frank-Olaf Schreyer – Syzygies, finite length modules and random curves I

1. A Gröbner basis proof of Hilbert's syzygy theorem
2. Petri's analysis of the generation of the ideal of a canonical embedded curve (1925).

Def A monomial order on $S = k[x_1, \dots, x_n]$ is a total order of the monomials satisfying

1) $x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma$
2) $x_i > 1$ for i (global)

$f = \sum f_\alpha x^\alpha$

$LT(f) = f_\alpha x^\alpha$ where x^α is the max of the set $\{x^\alpha / f_\alpha \neq 0\}$

Division Thm

Given $f_1, \dots, f_r \in S$, $g \in S$, $\exists!$ $g_1, \dots, g_r \in S$ and a remainder $h \in S$ s.t.

1) $g = g_1 f_1 + \dots + g_r f_r + h$

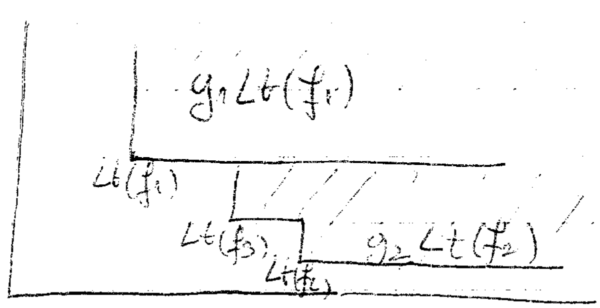
2a) for each i, no term of $g_i LT(f_i)$ is divisible by a LT of f_j for some $j < i$

2b) no term of h is divisible by $LT(f_j)$, any j.

Proof:

Clear for monomials

$g = g_1^{(1)} Lt(f_1) + \dots + g_r^{(1)} Lt(f_r) + h^{(1)}$ satisfying 2a) & 2b)



$g' = g - \left(\sum g_i^{(1)} f_i + h\right)$
$Lt(g') < Lt(g)$ and induction applies.

Remark 1) the remainder depends on the order of $f_1, \dots, f_r$

2) $g \in \langle f_1, \dots, f_r \rangle \not\Rightarrow h = 0$

Def $I \subset S$, $Lt(I) = \langle Lt(f) \mid f \in I \rangle$

Def $f_1, \dots, f_r$ form a $\underbrace{\text{Gröbner basis}}_{(GB)}$ if $Lt \langle f_1 \dots f_r \rangle = \langle Lt(f_1), \dots, Lt(f_r) \rangle$.

Consider the monomial ideals

$M_i = \cancel{Lt(f_i)} = \langle Lt(f_1), \dots, Lt(f_{i-1}) \rangle : Lt(f_i)$

Thm (Buchberger)

$f_1, \dots, f_r$ form a GB iff for each i and each minimal generator x^α of M_i, the remainder of $x^\alpha f_i$ divided by $f_1, \dots, f_r$ is zero.

Proof

$$x^\alpha f_i = g_1^{(i,\alpha)} f_1 + \dots + g_r^{(i,\alpha)} f_r + 0$$

so $G^{(i,\alpha)} = (-g_1^{(i,\alpha)}, \dots, \underset{\uparrow}{x^\alpha - g_i^{(i,\alpha)}}, \dots, -g_r^{(i,\alpha)})$
$\in \ker(S^r \longrightarrow S)$

On S^r we use the induced monomial order

$x^\alpha e_i > x^\beta e_j \iff x^\alpha Lt(f_i) > x^\beta Lt(f_j)$ or
$x^\alpha Lt(f_i) = x^\beta Lt(f_j)$ and $i > j$

In this order $Lt(G^{(i,\alpha)}) = x^\alpha e_i$

$g = \cancel{g_1} a_1 f_1 + \dots + a_r f_r$
divide $(a_1, \dots, a_r) \in S^r$ by $G^{(i,\alpha)}$'s $\Rightarrow$ remainder $(g_1, \dots, g_r)$ satisfies condition 2a)

$g = g_1 f_1 + \dots + g_r f_r$ since the $G^{(i,\alpha)}$ are syzygies
$Lt(g) = \max\{Lt(g_1)Lt(f_1), \dots, Lt(g_r)Lt(f_r)\} \in (Lt(f_1), \dots, Lt(f_r))$
since no cancellation occurs.

<u>Cor</u> The $G^{(i,\alpha)}$ form a GB of $\ker(S^r \to S)$.
<u>Proof</u>: A remainder $(g_1, \dots, g_r) \mapsto 0 \in S$ only if $(g_1, \dots, g_r) = 0$

<u>Rk</u> If we order $f_1, \dots, f_r$ defining the power in which the least occuring variable x_k of all $Lt(f_1), \dots, Lt(f_r)$ then $x_k \nmid x^\alpha$ a minimal generator $M_i \Rightarrow$ Hilbert's syzygy theorem.

2) C smooth projective curve of genus g non-hyperelliptic.
$\omega_1, \dots, \omega_g \in H^0(\omega_C)$

$$C \hookrightarrow \mathbb{P}^{g-1}$$
$$p \longmapsto (\omega_1(p) : \dots : \omega_g(p))$$

$S = k[x_1, \dots, x_g] \supset I_C$

<u>Thm</u> (Max Noether) S/I_C is CM.
<u>Thm</u> (Petri) I_C is generated by quadrics unless C is trigonal or isomorphic to a smooth plane quintic ($g = 6$).

$D = p_1 + \dots + p_d$ $\quad h^0 \omega_C(-D) = \operatorname{codim} \bar{D}$
$\Rightarrow \dim \bar{D} = \deg D - h^0 \mathcal{O}_C(D)$ $\quad$ lin. span of the points in $\mathbb{P}^{g-1}$

<u>Proof</u> Choose $p_1, \dots, p_g \in C$ general points and coordinates $x_1, \dots, x_g$ on $\mathbb{P}^{g-1}$ such that

$x_i(P_j) = 0$ for $i \neq j$

$D = p_1 + \dots + p_{g-2}$, $\Sigma \cap C = \{p_1, \dots, p_{g-2}\}$

$h^0 \mathcal{O}_C(D) = 1$.

$w_1, \dots, w_g \in H^0 \omega_C$ the corresponding bases $x_i \mapsto w_i$

$\underbrace{w_1^2, \dots, w_{g-2}^2}_{\text{does not vanish at } p_i} \notin H^0(\omega_C^{\otimes 2}(-D)) \subset H^0 \omega_C^{\otimes 2}$

represent a basis of $H^0\omega_C^{\otimes 2} / H^0\omega_C^{2}(-D)$.

$H^0 \omega_C(-D) = \langle x_{g-1}, x_g \rangle = \langle u, v \rangle$ is a basepoint free pencil

$0 \to \Lambda^2 H^0 \omega_C(-D) \otimes \mathcal{L} \to H^0 \omega_C(-D) \otimes \mathcal{L} \otimes \omega_C(-D) \to$

$\to \mathcal{L} \otimes \omega_C^{\otimes 2}(-2D) \to 0$ is exact (Koszul complex)

Take $\mathcal{L} = \mathcal{O}(D)$.

$0 \to H^0 \mathcal{O}(D) \to H^0 \omega_C(-1) \otimes H^0 \omega_C \to H^0 \omega_C^{\otimes 2}(-D) \to 0$

dim: 1 $\quad 2 \cdot g \quad$ $2g-1$

$(H^0 \omega^{\otimes 2}) = 3g-3)$

for $1 \leq i < j \leq g-2$, $w_i w_j \in H^0 \omega_C^{\otimes 2}(-D)$

$\Rightarrow \exists f_{ij} = x_i x_j - \sum_{k=i}^{g-2} a_{ij}^k x_k - b_{ij}$ $\leftarrow \binom{g-2}{2}$ elements

where a_{ij}^k are deg 1

b_{ij} are deg 2

We are missing $g-3$ cubic GB elements.

$\mathcal{L} = \omega_C$ yields

$0 \to H^0 \omega \to H^0 \omega_C(-D) \otimes H^0 \omega^{\otimes 2}(-D) \to H^0 \omega_C^{\otimes 3}(-2D)$

$g \qquad 2 \qquad 2g-1 \qquad 5g-5-(2(g-2))$

$\to 3g-2 \qquad 3g-1$

Consider the hyperplane $\alpha_i \in k[u,v]$ tangent at p_i to C. Then $\alpha_i w_i^2 \in H^0 \omega_C^{\otimes 3}(-2D)$.

None of them lies in the image, adjusting scalars $\alpha_i w_i^2 = \alpha_j w_j^2 \mod H^0(\omega_C^{\otimes 2}(-D))$.

$\Rightarrow \exists$ equations $G_{ij} = \alpha_i x_i^2 - \alpha_j x_j^2 + \sum$ linear in $x_1, \dots x_{g-2}$

$G_{ij} + G_{jk} = G_{ik}$

$x_i^2 x_{g-1}$, $i = 1, \dots, g-3$

$x_{g-2}^3 x_{g-1}$

$\rightsquigarrow$ GB

In the next lecture we'll finish the proof using Buchberger's Thm.