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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Idun Reiten

Talk Title: Cluster categories

Date: 09/04/12 Time: 9:00 am / pm (circle one)

List 6-12 key words for the talk: cluster category, quiver, tilting theory, Calabi-Yau

Please summarize the lecture in 5 or fewer sentances: This lecture describes cluster categories associated with finite acyclic quivers, cluster tilting objects and properties of cluster categories.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Idun Reiten - Cluster categories I

I Cluster categories via tilting theory [BMRRT = Buan, Marsh, Reineke, Reiten, Todorov], [BMR]

① The essential ingredients in the def. of cluster algebras [FZ]

$(n=3)$ $\mathcal{F} = \mathbb{Q}(x_1, x_2, x_3)$

Q = acyclic quiver with $n=3$ vertices [e.g $1 \to 2 \to 3$]

$(\{x_1, x_2, x_3\}, Q)$ is an initial seed

μ_1 μ_2 μ_3

$(\{x_1^*, x_2, x_3\}, \mu_1(Q))$ $(\{x_1, x_2^{**}, x_3\}, \mu_2(Q))$

Clusters : $\{x_1, x_2, x_3\}$, $\{x_1^*, x_2, x_3\}$ ....

Cluster var. : $x_1, x_2, x_3, x_1^*, \dots$

Seeds : $(\{x_1, x_2, x_3\}, Q), \dots$

Quiver mutation : $[\mu_2(1 \to 2 \to 3) = 1 \leftarrow 2 \leftarrow 3]$

Multiplication formula : $x_2 x_2^{**} = m_1 + m_2 = x_1 + x_3$

② Categorification

Find a "nice" category $\mathcal{C}$ with a lot of structure such that : (i) we have a collection $\mathcal{T}$ of basic objects

$T = T_1 \oplus \dots \oplus T_n$ with T_i indecomposable, distinct (no two isomorphic) summands such that $\forall i = 1, .., n$ there is a unique indecomposable $T_i^* \not\cong T_i$ in $\mathcal{C}$ such that $\underbrace{T/T_i \oplus T_i^*}_{= \mu_i(T)} \in \mathcal{T}$.

(ii) For T in $\mathcal{T}$, the quiver Q_T (= quiver of $\underbrace{\mathrm{End}(T)}_{\cong kQ/I}$) has no loops or 2-cycles

(iii) Want a commutative diagram

$$\begin{tikzcd}
T \arrow[r, mapsto, "\mu_i"] \arrow[d, mapsto] & \mu_i(T) \arrow[dr] & \\
Q_T \arrow[r, mapsto] & \mu_i(Q_T) \overset{\uparrow}{=} Q_{\mu_i(T)} &
\end{tikzcd}$$

i.e. we require this equality to hold

(iv) Related to $x_2 x_2^{**} = m_1 + m_2$

③ Tilting theory for kQ $\quad (1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$

Def: T in $\operatorname{mod} kQ = \operatorname{rep} Q$ is a tilting module if $\operatorname{Ext}^1_{kQ}(T,T) = 0$ and $|T| :=$ number of $\underbrace{\text{indecomposable}}_{\text{basic}}$ summands of T = # vertices in Q.

Thm [HU, U, RS]
If $T = T_1 \oplus \ldots \oplus T_m$ is a tilting kQ-module, T_i indec. then for $i \in \{1, \ldots, m\}$ there is at most one $T_i^* \not\cong T_i$ such that $T/T_i \oplus T_i^*$ is a tilting module.

Rk: it can happen that there is no T_i^*.

④ AR quivers

e.g. $Q = 1 \to 2 \to 3$, $\operatorname{mod} kQ$

$$\begin{tikzcd}
 & & \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \arrow[dr] & & \\
 & \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} \arrow[ur] \arrow[dr] & & \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
S_3 \arrow[ur] & & S_2 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & S_1 \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

Notation $\left[\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}\right]$ means S_3 is the socle of M.
S_2 is the socle of M/S_3
$k \xrightarrow{=} k \xrightarrow{=} k$ $\qquad$ S_1 is the socle of $(M/S_3)/S_2$

The diagram contains 3 almost split sequences

$$0 \to S_3 \longrightarrow \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} \longrightarrow S_2 \longrightarrow 0$$

X indec

⑤ Examples for tilting modules

$i=3$: $T = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} \oplus S_3 \xmapsto{\mu_3(T)} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} \oplus S_2$

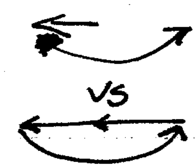
kQ (with $T \cong kQ$)

$$1 \longrightarrow 2 \longrightarrow 3 \quad \xmapsto{\mu_3} \quad 1 \longrightarrow 2 \longleftarrow 3$$

check: commutes

$i=2$: replace $\begin{pmatrix} S_2 \\ S_3 \end{pmatrix}$ by S_1

but we get different quivers

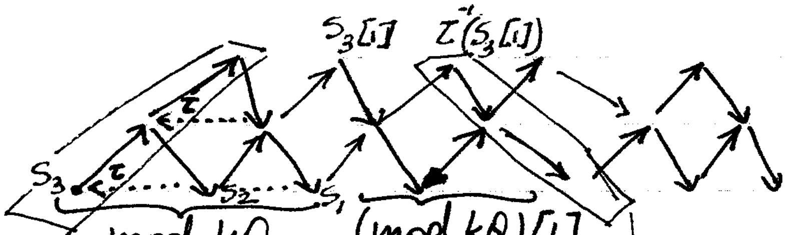


$i=1$: $\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$ has no complement

$\Rightarrow$ need more objects and more maps

⑥ The cluster category

Consider $\operatorname{mod} kQ \subset D^b(kQ)$



AR-quiver for $D^b(kQ)$

$$\boxed{\mathcal{C}_Q \overset{\text{def}}{=} D^b(kQ) / \tau^{-1}[1]}$$

Thm (Keller) $\mathcal{C}_Q$ is triangulated.

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}_Q}(S_1, S_3) &= \operatorname{Hom}_{kQ}(S_1, S_3) \oplus \operatorname{Hom}_{kQ}(S_1, \tau^{-1}(S_3[1])) \oplus \dots \\
&= 0 \oplus \operatorname{Hom}_{kQ}(S_1, \tau^{-1}(S_3[1])) \oplus 0 \oplus 0 \oplus \dots \\
&= \operatorname{Hom}_{kQ}(S_1, S_2[1]) \\
&= \operatorname{Ext}^1_{kQ}(S_1, S_2) \neq 0
\end{aligned}$$

$0 \to S_2 \to \begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix} \to S_1 \to 0$ not split

Cluster tilting object: $\operatorname{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$, $|T| = n$.

Def: Almost split triangle:

$$A \longrightarrow B \longrightarrow C \xrightarrow{\mu} A[1]$$

- A, B indecomposable
- $\mu \neq 0$
-

$$\begin{tikzcd} & & X \arrow[dll, dotted] \arrow[d, "\ncong"] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

X indecomposable