Vector bundles and ideal closure operations

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Contents

1. Lecture

Linear equations, forcing algebras and ideal closure operations

1.1. Systems of linear equations.

We start with some linear algebra. Let K be a field. We consider a system of linear homogeneous equations over K,

$$f_{11}t_1 + \ldots + f_{1n}t_n = 0\,,$$

$$f_{21}t_1 + \ldots + f_{2n}t_n = 0\,,$$

$$\vdots$$

$$f_{m1}t_1 + \ldots + f_{mn}t_n = 0\,,$$

where the f_{ij} are elements in K. The solution set to this system of homogeneous equations is a vector space V over K, its dimension is $n - \mathrm{rk}(A)$, where $A = (f_{ij})_{ij}$ is the matrix given by these elements. Additional elements $f_1, \ldots, f_m \in K$ give rise to the system of inhomogeneous linear equations,

$$f_{11}t_1 + \ldots + f_{1n}t_n = f_1\,,$$

$$f_{21}t_1 + \ldots + f_{2n}t_n = f_2\,,$$

$$\vdots$$

$$f_{m1}t_1 + \ldots + f_{mn}t_n = f_m\,.$$

The solution set T of this inhomogeneous system may be empty, but nevertheless it is tightly related to the solution space of the homogeneous system. First of all, there exists an action

$$V \times T \longrightarrow T,\ (v,t) \longmapsto v + t,$$

because the sum of a solution of the homogeneous system and a solution of the inhomogeneous system is again a solution of the inhomogeneous system. This action is a group action of the group $(V, +, 0)$ on the set T. Moreover, if we fix one solution $t_0 \in T$ (supposing that at least one solution exists), then there exists a bijection

$$V \longrightarrow T,\ v \longmapsto v + t_0.$$

So T can be identified with the vector space V, however not in a canonical way. The group V acts simply transitive on T.

Suppose now that X is a geometric object (a topological space, a manifold, a variety, the spectrum of a ring) and that instead of elements in the field K we have functions

$$f_{ij} : X \longrightarrow K$$

on X (which are continuous, or differentiable, or algebraic). We form the Matrix of functions $A = (f_{ij})_{ij}$, which yields for every point $P \in X$ a matrix $A(P)$ over K. Then we get from these data the space

$$V = \left\{ (P, t_1, \ldots, t_n) |\, A(P) \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = 0 \right\} \subseteq X \times K^n$$

together with the projection to X. For a fixed point $P \in X$, the fiber of V over P is the solution space to the corresponding homogeneous system of linear equations given by inserting P. In particular, all fibers of the map

$$V \longrightarrow X,$$

are vector spaces (maybe of non-constant dimension). This vector space structures yield an addition

$$V \times_X V \longrightarrow V,\ (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n)$$

(only points in the same fiber can be added). The mapping

$$X \longrightarrow V,\ P \longmapsto (P; 0, \ldots, 0)$$

is called the zero-section.

Suppose now that there are additionally functions

$$f_1, \ldots, f_m : X \longrightarrow K$$

given. Then we can form the set

$$T = \left\{ (P, t_1, \ldots, t_n) |\, A(P) \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = \begin{pmatrix} f_1(P) \\ \\ f_n(P) \end{pmatrix} \right\} \subseteq X \times K^n$$

with the mapping to X. Again, every fiber of T over a point $P \in X$ is the solution set to the system of inhomogeneous linear equations which arises by inserting P. The actions of the fibers V_P on T_P (coming from linear algebra) extend to an action

$$V \times_X T \longrightarrow T,\ (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n).$$

Also, if a (continuous, differentiable, algebraic) map

$$s : X \longrightarrow T$$

with $s(P) \in T_P$ exists, then we can construct an (continuous, differentiable, algebraic) isomorphism between V and T. However, different from the situation in linear algebra (which corresponds to the situation where X is just one point), such a section does rarely exist.

These objects T have new and sometimes difficult global properties which we try to understand in these lectures. We will work mainly in an algebraic setting and restrict to the situation where just one equation

$$f_1T_1 + \ldots + f_nT_n = f$$

is given. Then in the homogeneous case ($f = 0$) the fibers are vector spaces of dimension $n-1$ or n, and the later holds exactly for the points $P \in X$ where $f_1(P) = \ldots = f_n(P) = 0$. In the inhomogeneous case the fibers are either empty or of dimension $n-1$ or n. We give some typical examples.

Example 1.1. We consider the line ($X = \mathbb{A}^1_K$) (or $X = K, \mathbb{R}, \mathbb{C}$ etc.) with the (identical) function x. For $f_1 = x$ and $f = 0$, i.e. for the equation $xt = 0$, the geometric object V consists of a horizontal line (corresponding to the zero-solution) and a vertical line over $x = 0$. So all fibers except one are zero-dimensional vector spaces. For the equation $0t = x$, V consists of one vertical line, almost all fibers are empty. For the equation $xt = 1$, V is a hyperbola, and all fibers are zero-dimensional with the exception that the fiber over $x = 0$ is empty.

Example 1.2. Let X denote a plane ($K^2, \mathbb{R}^2, \mathbb{A}^2_K$) with coordinate functions x and y. We consider a linear equation of type

$$x^at_1 + y^bt_2 = x^cy^d\,.$$

The fiber of the solution set T over a point $\neq (0,0)$ is onedimensional, whereas the fiber over $(0,0)$ has dimension two (for $a, b, c, d \geq 1$). Many properties of T depend on these four exponents.

In (most of) these example we can observe the following behavior. On an open subset, the dimension of the fibers is constant and equals $n-1$, whereas the fiber over some special points degenerates to an n-dimensional solution set (or becomes empty).

1.2. **Forcing algebras.**

We describe now the algebraic setting of systems of linear equations depending on a base space. For a commutative ring R, its spectrum $X = \mathrm{Spec}\,(R)$ is a topological space on which the ring elements can be considered as functions. The value of $f \in R$ at a prime ideal $P \in \mathrm{Spec}\,(R)$ is just the image of f under the morphism $R \to R/P \to \kappa(P) = Q(R/P)$. In this interpretation, a ring element is a function with values in different fields. Suppose that R contains a field K. Then an element $f \in R$ gives rise to the ring homomorphism

$$K[Y] \longrightarrow R,\, Y \longmapsto f,$$

which itself gives rise to a scheme morphism

$$\mathrm{Spec}\,(R) \longrightarrow \mathrm{Spec}\,(K[Y]) \cong \mathbb{A}^1_K.$$

This is another way to consider f as a function on $\mathrm{Spec}\,(R)$.

Definition 1.3. Let R be a commutative ring and let $f_1, \ldots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

is called the *forcing algebra* of these elements (or these data).

The forcing algebra B forces f to lie inside the extended ideal $(f_1, \ldots, f_n)B$ (hence the name) For every R-algebra S such that $f \in (f_1, \ldots, f_n)S$ there exists a (non unique) ring homomorphism $B \to S$ by sending T_i to the coefficient $s_i \in S$ in an expression $f = s_1f_1 + \ldots + s_nf_n$.

The forcing algebra induces the spectrum morphism

$$\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(R).$$

Over a point $P \in X = \operatorname{Spec}(R)$, the fiber of this morphism is given by

$$\operatorname{Spec}(B \otimes_R \kappa(P)),$$

and we can write

$$B \otimes_R \kappa(P) = \kappa(P)[T_1, \ldots, T_n]/(f_1(P)T_1 + \ldots + f_n(P)T_n - f(P)),$$

where $f_i(P)$ means the evaluation of the f_i in the residue class field. Hence the $\kappa(P)$-points in the fiber are exactly the solution to the inhomogeneous linear equation $f_1(P)T_1 + \ldots + f_n(P)T_n = f(P)$. In particular, all the fibers are (empty or) affine spaces.

1.3. **Forcing algebras and closure operations.**

Let R denote a commutative ring and let $I = (f_1, \ldots, f_n)$ be an ideal. Let $f \in R$ and let

$$B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

be the corresponding forcing algebra and

$$\varphi : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(R)$$

the corresponding spectrum morphism. How are properties of φ (or of the R-algebra B) related to certain ideal closure operations?

We start with some examples. The element f belongs to the ideal I if and only if we can write $f = r_1f_1 + \ldots + r_nf_n$. By the universal property of the forcing algebra this means that there exists an R-algebra-homomorphism

$$B \longrightarrow R,$$

hence $f \in I$ holds if and only if φ admits a scheme section. This is also equivalent to

$$R \longrightarrow B$$

admitting an R-module section or B being a pure R-algebra (so for forcing algebras properties might be equivalent which are not equivalent for arbitrary algebras).

We have a look at the radical of the ideal I,

$$\operatorname{rad}(I) = \left\{ f \in R |\, f^k \in I \text{ for some } k \right\} .$$

As this is quite a coarse closure operation we should expect that this corresponds to a quite coarse property of the morphism φ as well. Indeed, it is true that $f \in \operatorname{rad}(I)$ if and only if φ is surjective. This and the interpretation of other closure operations in terms of forcing algebras will be discussed in the tutorial session and in the next lecture.

1.4. **Geometric vector bundles.**

We have seen that the fibers of the spectrum of a forcing algebra are (empty or) affine spaces. However, this is not only fiberwise true, but more general: If we localize the forcing algebra at f_i we get

$$(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f))_{f_i} \cong R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n],$$

since we can write

$$T_i = -\sum_{j \neq i} \frac{f_j}{f_i} T_j + \frac{f}{f_i} .$$

So over every $D(f_i)$ the spectrum of the forcing algebra is an $(n-1)$-dimensional affine space over the base. So locally, restricted to $D(f_i)$, we have isomorphisms

$$T|_{D(f_i)} \cong D(f_i) \times \mathbb{A}^{n-1} .$$

On the intersections $D(f_i) \cap D(f_j)$ we get two identifications with affine space, and the transition morphisms are linear if $f = 0$, but only affine-linear in general (because of the translation with $\frac{f}{f_i}$).

So the forcing algebra has locally the form $R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]$ and its spectrum $\operatorname{Spec}(B)$ has locally the form $D(f_i) \times \mathbb{A}_K^{n-1}$. This description holds on the union $U = \bigcup_{i=1}^n D(f_i)$. Moreover, in the homogeneous case (f=0) the transition mappings are linear. Hence $V|_U$ is a geometric vector bundle according to the following definition.

Definition 1.4. Let X denote a scheme. A scheme

$$p : V \longrightarrow X$$

is called a *geometric vector bundle* of rank r over X if there exists an open covering $X = \bigcup_{i \in I} U_i$ and U_i-isomorphisms

$$\psi_i : U_i \times \mathbb{A}^r = \mathbb{A}^r_{U_i} \longrightarrow V|_{U_i} = p^{-1}(U_i)$$

such that for every open affine subset $U \subseteq U_i \cap U_j$ the transition mappings

$$\psi_j^{-1} \circ \psi_i : \mathbb{A}^r_{U_i}|_U \longrightarrow \mathbb{A}^r_{U_j}|_U$$

are linear automorphisms, i.e. they are induced by an automorphism of the polynomial ring $\Gamma(U, \mathcal{O}_X)[T_1, \ldots, T_r]$ given by $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$.

Here we can restrict always to affine open coverings. If X is separated then the intersection of two affine open subschemes is again affine and then it is enough to check the condition on the intersection. The trivial bundle of rank r is the r-dimensional affine space $\mathbb{A}^r_X$ over X, and locally every vector bundle looks like this. Many properties of an affine space are enjoyed by general vector bundles. For example, in the affine space we have the natural addition

$$+ : \mathbb{A}^r_U \times_U \mathbb{A}^r_U \longrightarrow \mathbb{A}^r_U,\ (v_1, \ldots, v_r, w_1, \ldots, w_r) \longmapsto (v_1 + w_1, \ldots, v_r + w_r),$$

and this carries over to a vector bundle, that is, we have an addition

$$\alpha : V \times_X V \longrightarrow V.$$

The reason for this is that the isomorphisms occurring in the definition of a geometric vector bundle are linear, hence the addition on $V|_U$ coming from an isomorphism with some affine space over U is independent of the choosen isomorphism. For the same reason there is a unique closed subscheme of V called the *zero-section* which is locally defined to be $0 \times U \subseteq \mathbb{A}^r_U$. Also, the multiplication by a scalar, i.e. the mapping

$$\cdot : \mathbb{A}_U \times_U \mathbb{A}^r_U \longrightarrow \mathbb{A}^r_U,\ (s, v_1, \ldots, v_r) \longmapsto (sv_1, \ldots, sv_r),$$

carries over to a scalar multiplication

$$\cdot : \mathbb{A}_X \times_X V \longrightarrow V.$$

In particular, for every point $P \in X$ the fiber $V_P = V \times_X P$ is an affine space over $\kappa(P)$.

For a geometric vector bundle $p : V \to X$ and an open subset $U \subseteq X$ one sets

$$\Gamma(U, V) = \{ s : U \to V|_U \,|\, p \circ s = \mathrm{id}_U \}\ ,$$

so this is the set of sections in V over U. This gives in fact for every scheme over X a set-valued sheaf. Because of the observations just mentioned, these sections can also be added and multiplied by elements in the structure sheaf, and so we get for every vector bundle a locally free sheaf, which is free on the open subsets where the vector bundle is trivial.

Definition 1.5. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called *locally free* of rank r, if there exists an open covering $X = \bigcup_{i \in I} U_i$ and $\mathcal{O}_{U_i}$-module-isomorphisms $\mathcal{F}|_{U_i} \cong \mathcal{O}^r|_{U_i}$ for every $i \in I$.

Vector bundles and locally free sheaves are essentially the same objects.

Theorem 1.6. *Let X denote a scheme. Then the category of locally free sheaves on X and the category of geometric vector bundles on X are equivalent. A geometric vector bundle $V \to X$ corresponds to the sheaf of its sections, and a locally free sheaf $\mathcal{F}$ corresponds to the (relative) Spectrum of the symmetric algebra of the dual module $\mathcal{F}^*$.*

The free sheaf of rank r corresponds to the affine space $\mathbb{A}^r_X$ over X.

As the solution vector space of a system of homogeneous linear equations acts on the solution set of a system of inhomogeneous linear equations, the spectrum of a homogeneous forcing algebra acts on the spectrum of an inhomogeneous forcing algebra. This action is given by

$$\text{Spec}\,(A)\times\text{Spec}\,(B) \longrightarrow \text{Spec}\,(B),\ (t_1,\ldots,t_n,s_1,\ldots,s_n) \longmapsto (t_1+s_1,\ldots,t_n+s_n).$$

On the ring level this map is induced by $S_i \mapsto S_i + T_i$.

2. Lecture

2.1. **Torsors of vector bundles.**

We have seen that $V = \text{Spec}\,(R[T_1,\ldots,T_n]/(f_1T_1+\ldots+f_nT_n))$ acts on the spectrum of a forcing algebra $T = \text{Spec}\,(R[T_1,\ldots,T_n]/(f_1T_1+\ldots+f_nT_n+f))$ by addition. The restriction of V to U is a vector bundle, and T restricted to U becomes a V-torsor.

Definition 2.1. Let V denote a geometric vector bundle over a scheme X. A scheme $T \to X$ together with an action

$$\beta : V \times_X T \longrightarrow T$$

is called a geometric (Zariski)-*torsor* for V (or a V-*principal fiber bundle* or a *principal homogeneous space*) if there exists an open covering $X = \bigcup_{i\in I} U_i$ and isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

such that the diagrams (we set $U = U_i$ and $\varphi = \varphi_i$)

$$\begin{array}{ccc} V|_U \times_U T|_U & \xrightarrow{\beta} & T|_U \\ \downarrow & & \downarrow \\ V|_U \times_U V|_U & \xrightarrow{\alpha} & V|_U \end{array}$$

commute.

The torsors of vector bundles can be classified in the following way.

Proposition 2.2. *Let X denote a Noetherian separated scheme and let*

$$p : V \longrightarrow X$$

denote a geometric vector bundle on X with sheaf of sections $\mathcal{S}$. Then there exists a correspondence between first cohomology classes $c \in H^1(X,\mathcal{S})$ and geometric V-torsors.

Beweis. We will describe this correspondence. Let T denote a V-torsor. Then there exists by definition an open covering $X = \bigcup_{i\in I} U_i$ such that there exists isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

which are compatible with the action of $V|_{U_i}$ on itself. The isomorphisms φ_i induce automorphisms

$$\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j}.$$

These automorphisms are compatible with the action of V on itself, and this means that they are of the form

$$\psi_{ij} = \mathrm{Id}_V\,|_{U_i \cap U_j} + s_{ij}$$

with suitable sections $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$. This family defines a Cech-cocycle for the covering and gives therefore a cohomology class in $H^1(X, \mathcal{S})$. For the reverse direction, suppose that the cohomology class $c \in H^1(X, \mathcal{S})$ is represented by a Cech-cocycle $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$ for an open covering $X = \bigcup_{i \in I} U_i$. Set $T_i := V|_{U_i}$. We take the morphisms

$$\psi_{ij} : T_i|_{U_i \cap U_j} = V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$$

given by $\psi_{ij} := \mathrm{Id}_V\,|_{U_i \cap U_j} + s_{ij}$ to glue the T_i together to a scheme T over X. This is possible since the cocycle condition guarantees the glueing condition for schemes (EGA I, 0, 4.1.7). The action of $T_i = V|_{U_i}$ on itself glues also together to give an action on T. □

It follows immediately that for an affine scheme (i.e. a scheme of type $\mathrm{Spec}\,(R)$) there are no non-trivial torsor for any vector bundle. There will however be in general many non-trivial torsors on the punctured spectrum (and on a projective variety).

2.2. **Forcing algebras and induced torsors.**

As T_U is a V_U-torsor, and as every V-torsor is represented by a unique cohomology class, there should be a natural cohomology class coming from the forcing data. To see this, let R be a noetherian ring and $I = (f_1, \ldots, f_n)$ be an ideal. Then on $U = D(I)$ we have the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

An element $f \in R$ defines an element $f \in \Gamma(U, \mathcal{O}_U)$ and hence a cohomology class $\delta(f) \in H^1(U, \mathrm{Syz}\,(f_1, \ldots, f_n))$. Hence f defines in fact a $\mathrm{Syz}\,(f_1, \ldots, f_n)$-torsor over U. We will see that this torsor is induced by the forcing algebra given by $f_1, \ldots, f_n$ and f.

Theorem 2.3. *Let R denote a noetherian ring, let $I = (f_1, \ldots, f_n)$ denote an ideal and let $f \in R$ be another element. Let $c \in H^1(D(I), \mathrm{Syz}\,(f_1, \ldots, f_n))$ be the corresponding cohomology class and let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$ denote the forcing algebra for these data. Then the scheme $\mathrm{Spec}\,(B)|_{D(I)}$ together with the natural action of the syzygy bundle on it is isomorphic to the torsor given by c.*

Beweis. We compute the cohomology class $\delta(f) \in H^1(U, \mathrm{Syz}\,(f_1, \ldots, f_n))$ and the cohomology class given by the forcing algebra. For the first computation we look at the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

On $D(f_i)$, the element f is the image of $(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0)$ (the non-zero entry is at the ith place). The cohomology class is therefore represented by the family of differences

$$(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0, -\frac{f}{f_j}, 0, \ldots, 0) \in \Gamma(D(f_i) \cap D(f_j), \mathrm{Syz}\,(f_1, \ldots, f_n))\,.$$

On the other hand, there are isomorphisms

$$V|_{D(f_i)} \longrightarrow T|_{D(f_i)},\ (s_1, \ldots, s_n) \longmapsto (s_1, \ldots, s_{i-1}, s_i + \frac{f}{f_i}, s_{i+1}, \ldots, s_n).$$

The difference of two such isomorphisms on $D(f_i f_j)$ is the same as before. □

Example 2.4. Let $(R, \mathfrak{m})$ denote a two-dimensional normal local noetherian domain and let f and g be two parameters in R. On $U = D(\mathfrak{m})$ we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_U \cong \mathrm{Syz}\,(f, g) \longrightarrow \mathcal{O}_U^2 \xrightarrow{f,g} \mathcal{O}_U \longrightarrow 0$$

and its corresponding long exact sequence of cohomology,

$$0 \longrightarrow R \longrightarrow R^2 \xrightarrow{f,g} R \xrightarrow{\delta} H^1(U, \mathcal{O}) \longrightarrow \ldots\,.$$

The connecting homomorphisms δ sends an element $h \in R$ to $\frac{h}{fg}$. The torsor given by such a cohomology class $c = \frac{h}{fg} \in H^1(U, \mathcal{O}_X)$ can be realized by the forcing algebra

$$R[T_1, T_2]/(fT_1 + gT_2 - h)\,.$$

Note that different forcing algebras may give the same torsor, because the torsor depends only on the spectrum of the forcing algebra restricted to the punctured spectrum of R. For example, the cohomology class $\frac{1}{fg} = \frac{fg}{f^2g^2}$ defines one torsor, but the two quotients yield the two forcing algebras $R[T_1, T_2]/(fT_1 + gT_2 - 1)$ and $R[T_1, T_2]/(f^2T_1 + g^2T_2 - fg)$, which are quite different. The fiber over the maximal ideal of the first one is empty, whereas the fiber over the maximal ideal of the second one is a plane.

If R is regular, say $R = K[X, Y]$ (or the localization of this at (X, Y) or the corresponding power series ring) then the first cohomology classes are K-linear combinations of $\frac{1}{x^i y^j}$, $i, j \geq 1$. They are realized by the forcing algebras $K[X, Y, T_1, T_2]/(X^iT_1 + Y^jT_2 - 1)$. Since the fiber over the maximal ideal is empty, the spectrum of the forcing algebra equals the torsor. Or, the other way round, the torsor is itself an affine scheme.

The closure operations we have considered in the second lecture can be characterized by some property of the forcing algebra. However, they can not be characterized by a property of the corresponding torsor alone. For example, for $R = K[X, Y]$, we may write

$$\frac{1}{XY} = \frac{X}{X^2Y} = \frac{XY}{X^2Y^2} = \frac{X^2Y^2}{X^3Y^3},$$

so the torsors given by the forcing algebras

$$R[T_1, T_2]/(XT_1 + YT_2 + 1), R[T_1, T_2]/(X^2T_1 + YT_2 + X),$$
$$R[T_1, T_2]/(X^2T_1 + Y^2T_2 + XY) \text{ and } R[T_1, T_2]/(X^3T_1 + Y^3T_2 + X^2Y^2)$$

are all the same (the restriction over $D(X, Y)$), but their global properties are quite different. We have a non-surjection, a surjective non submersion, a submersion which does not admit (for $K = \mathbb{C}$) a continuous section and a map which admits a continuous section.

We deal now with closure operations which depend only on the torsor which the forcing algebra defines, so they only depend on the cohomology class of the forcing data inside the syzygy bundle. Our main example is tight closure, a theory developed by Hochster and Huneke, and related closure operations like solid closure and plus closure.

2.3. **Tight closure and solid closure.**

Let R be a noetherian domain of positive characteristic, let

$$F : R \longrightarrow R,\ f \longmapsto f^p,$$

be the *Frobenius homomorphism*, and

$$F^e : R \longrightarrow R,\ f \longmapsto f^q,\ q = p^e\,,$$

its eth iteration. Let I be an ideal and set

$$I^{[q]} = \text{ extended ideal of } I \text{ under } F^e$$

Then define the *tight closure* of I to be the ideal

$$I^* := \{f \in R : \text{ there exists } z \neq 0 \text{ such that } zf^q \in I^{[q]} \text{ for all } q = p^e\}\,.$$

The element f defines the cohomology class $c \in H^1(D(I), \mathrm{Syz}\,(f_1, \ldots, f_n))$. Suppose that R is normal and that I has height at least 2 (think of a local normal domain of dimension at least 2 and an $\mathfrak{m}$-primary ideal I). Then the eth Frobenius pull-back of the cohomology class is

$$F^{e*}(c) \in H^1(D(I), F^{e*}(\mathrm{Syz}\,(f_1, \ldots, f_n)) \cong H^1(D(I), \mathrm{Syz}\,(f_1^q, \ldots, f_n^q))$$

($q = p^e$) and this is the cohomology class corresponding to f^q. By the height assumption, $zF^{e*}(c) = 0$ if and only if $zf^q \in (f_1^q, \ldots, f_n^q)$, and if this holds for all e then $f \in I^*$ by definition. This shows already that tight closure under the given conditions does only depend on the cohomology class.

This is also a consequence of the following theorem of Hochster which gives a characterization of tight closure in terms of forcing algebra and local cohomology.

Theorem 2.5. *Let R be a normal excellent local domain with maximal ideal $\mathfrak{m}$ over a field of positive characteristic. Let $f_1, \ldots, f_n$ generate an $\mathfrak{m}$-primary ideal I and let f be another element in R. Then $f \in I^*$ if and only if*

$$H^{\dim(R)}_{\mathfrak{m}}(B) \neq 0\,,$$

where $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ denotes the forcing algebra of these elements.

If the dimension d is at least two, then

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}B}(B) \cong H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B)\,.$$

This means that we have to look at the cohomological properties of the complement of the exceptional fiber over the closed point, i.e. the torsor given by these data. If the dimension is two, then we have to look whether the first cohomology of the structure sheaf vanishes. This is true (by Serre's cohomological criterion for affineness) if and only if the open subset $D(\mathfrak{m}B)$ is an *affine scheme* (the spectrum of a ring).

The right hand side of this equivalence - the non-vanishing of the top-dimensional local cohomology - is independent of any characteristic assumption, and can be taken as the basis for the definition of another closure operation, called *solid closure*. So the theorem above says that in positive characteristic tight closure and solid closure coincide. There is also a definition of tight closure for algebras over a field of characteristic 0 by reduction to positive characteristic.

An important property of tight closure is that it is trivial for regular rings, i.e. $I^* = I$ for every ideal I. This rests upon Kunz's theorem saying that the Frobenius homomorphism for regular rings is flat. This property implies the following cohomological property of torsors.

Corollary 2.6. Let $(R, \mathfrak{m})$ denote a regular local ring of dimension d and of positive characteristic, let $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal and $f \in R$ an element with $f \notin I$. Let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ be the corresponding forcing algebra. Then for the extended ideal $\mathfrak{m}B$ we have

$$H^d_{\mathfrak{m}B}(B) \;=\; H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B) \;=\; 0.$$

Beweis. This follows from Theorem 2.5 and $f \notin I^*$. □

2.4. **Plus closure.**

For an ideal $I \subseteq R$ in a domain R define

$I^+ = \{f \in R :$ there exists a finite domain extension $R \subseteq T$ such that $f \in IT\}$.

Equivalent: let R^+ be the *absolute integral closure* of R. This is the integral closure of R in an algebraic closure of the quotient field $Q(R)$ (first considered by Artin). Then

$$f \in I^+ \text{ if and only if } f \in IR^+ .$$

The plus closure commutes with localization.

We also have the inclusion $I^+ \subseteq I^*$. Here the question arises:

Question: Is $I^+ = I^*$?

This question is known as the *tantalizing question* in tight closure theory.

In terms of forcing algebras and their torsors, the containment inside the plus closure means that there exists a d-dimensional closed subscheme inside the torsor which meets the exceptional fiber (the fiber over the maximal ideal) in one point, and this means that the superheight of the extended ideal is d. In this case the local cohomological dimension of the torsor must be d as well, since it contains a closed subscheme with this cohomological dimension. So also the plus closure depends only on the torsor.

3. Lecture

In the last lectures we will continue with the question when are the torsors given by a forcing algebras over a two-dimensional ring affine? We will look at the graded situation to be able to work on the corresponding projective curve.

In particular we want to address the following questions

(1) Is there a procedure to decide whether the torsor is affine?
(2) Is it non-affine if and only if there exists a geometric reason for it not to be affine (because the superheight is too large)?
(3) How does the affineness vary in an arithmetic family, when we vary the prime characteristic?
(4) How does the affineness vary in a geometric family, when we vary the base ring?

In terms of tight closure, these questions are directly related to the tantalizing question of tight closure (is it the same as plus closure), the dependence of tight closure on the characteristic and the localization problem of tight closure.

3.1. **Geometric interpretation in dimension two.**

We will restrict now to the two-dimensional homogeneous case in order to work on the corresponding projective curve. We want to find an object over the curve which corresponds to the forcing algebra or its induced torsor.

Let R be a two-dimensional standard-graded normal domain over an algebraically closed field K. Let $C = \operatorname{Proj} R$ be the corresponding smooth

projective curve and let

$$I = (f_1, \ldots, f_n)$$

be an R_+-primary homogeneous ideal with generators of degrees $d_1, \ldots, d_n$. Then we get on C the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0\,.$$

Here $\mathrm{Syz}\,(f_1, \ldots, f_n)(m)$ is a vector bundle, called the *syzygy bundle*, of rank $n-1$ and of degree

$$((n-1)m - \sum_{i=1}^{n} d_i)\deg(C)\,.$$

Thus a homogeneous element f of degree m defines a cohomology class $\delta(f) \in H^1(C, \mathrm{Syz}\,(f_1, \ldots, f_n)(m))$, so this defines a torsor over the projective curve. We mention an alternative description of the torsor corresponding to a first cohomology class in a locally free sheaf which is better suited for the projective situation.

Remark 3.1. Let $\mathcal{S}$ denote a locally free sheaf on a scheme X. For a cohomology class $c \in H^1(X, \mathcal{S})$ one can construct a geometric object: Because of $H^1(X, \mathcal{S}) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{S})$, the class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

This extension is such that under the connecting homomorphism of cohomology, $1 \in \Gamma(X, \mathcal{O}_X)$ is sent to $c \in H^1(X, \mathcal{S})$. The extension yields projective subbundles

$$\mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}(\mathcal{S}'^\vee)\,.$$

If V is the corresponding geometric vector bundle, one may think of $\mathbb{P}(\mathcal{S}^\vee)$ as $\mathbb{P}(V)$ which consists for every base point $x \in X$ of all the lines in the fiber V_x passing through the origin. The projective subbundle $\mathbb{P}(V)$ has codimension one inside $\mathbb{P}(V')$, for every point it is a projective space lying (linearly) inside a projective space of one dimension higher. The complement is then over every point an affine space. One can show that the global complement

$$T = \mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$$

is another model for the torsor given by the cohomology class. The advantage of this viewpoint is that we may work, in particular when X is projective, in an entirely projective setting.

3.2. **Semistability of vector bundles.**

In the situation of a forcing algebra for homogeneous elements, this torsor T can also be obtained as Proj B, where B is the (not necessarily positively) graded forcing algebra. In particular, it follows that the containment $f \in I^*$ is equivalent to the property that T is not an affine variety. For this properties,

positivity (ampleness) properties of the syzygy bundle are crucial. We need the concept of (Mumford) semistability.

Definition 3.2. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C. It is called *semistable*, if $\mu(\mathcal{T}) = \frac{\deg(\mathcal{T})}{\mathrm{rk}(\mathcal{T})} \leq \frac{\deg(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} = \mu(\mathcal{S})$ for all subbundles $\mathcal{T}$.

Suppose that the base field has positive characteristic $p > 0$. Then $\mathcal{S}$ is called *strongly semistable*, if all (absolute) Frobenius pull-backs $F^{e*}(\mathcal{S})$ are semistable.

An important property of a semistable bundle of negative degree is that it can not have any global section $\neq 0$. Note that a semistable vector bundle need not be strongly semistable, the following is probably the simplest example.

Example 3.3. Let C be the smooth Fermat quartic given by $x^4 + y^4 + z^4$ and consider on it the syzygy bundle $\mathrm{Syz}\,(x, y, z)$ (which is also the restricted cotangent bundle from the projective plane). This bundle is semistable. Suppose that the characteristic is 3. Then its Frobenius pull-back is $\mathrm{Syz}\,(x^3, y^3, z^3)$. The curve equation gives a global nontrivial section of this bundle of total degree 4. But the degree of $\mathrm{Syz}\,(x^3, y^3, z^3)(4)$ is negative, hence it can not be semistable anymore.

For a strongly semistable vector bundle $\mathcal{S}$ on C and a cohomology class $c \in H^1(C, \mathcal{S})$ with corresponding torsor we obtain the following affineness criterion.

Theorem 3.4. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a strongly semistable vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Then the torsor $T(c)$ is an affine scheme if and only if* $\deg(\mathcal{S}) < 0$ *and $c \neq 0$ ($F^e(c) \neq 0$ for all e in positive characteristic).*

This result rests on the ampleness of $\mathcal{S}'^{\vee}$ occuring in the dual exact sequence $0 \to \mathcal{O}_C \to \mathcal{S}'^{\vee} \to \mathcal{S}^{\vee} \to 0$ given by c (work of Hartshorne and Gieseker). It implies for a strongly semistable syzygy bundles the following *degree formula* for tight closure.

Theorem 3.5. *Suppose that* $\mathrm{Syz}\,(f_1, \ldots, f_n)$ *is strongly semistable. Then*

$$R_m \subseteq I^* \text{ for } m \geq \frac{\sum d_i}{n-1} \text{ and (for almost all prime numbers)}$$

$$R_m \cap I^* \subseteq I \text{ for } m < \frac{\sum d_i}{n-1}.$$

We indicate the proof of the inclusion result. The degree condition implies that $c = \delta(f) \in H^1(C, \mathcal{S})$ is such that $\mathcal{S} = \mathrm{Syz}\,(f_1, \ldots, f_n)(m)$ has nonnegative degree. Then also all Frobenius pull-backs $F^*(\mathcal{S})$ have nonnegative degree. Let $\mathcal{L} = \mathcal{O}(k)$ be a twist of the tautological line bundle on C such

that its degree is larger than the degree of ω_C^{-1}, the dual of the canonical sheaf. Let $z \in H^0(Y, \mathcal{L})$ be a non-zero element. Then $zF^{e*}(c) \in H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L})$, and by Serre duality we have

$$H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L}) \cong H^0(F^{e*}(\mathcal{S}^*) \otimes \mathcal{L}^{-1} \otimes \omega_C)^\vee .$$

On the right hand side we have a semistable sheaf of negative degree, which can not have a nontrivial section. Hence $zF^{e*} = 0$ and therefore f belongs to the tight closure.

3.3. **Harder-Narasimhan filtration.**

In general, there exists an exact criterion depending on c and the *strong Harder-Narasimhan filtration* of $\mathcal{S}$. For this we give the definition of the Harder-Narasimhan filtration.

Definition 3.6. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C over an algebraically closed field K. Then the (uniquely determined) filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

of subbundles such that all quotient bundles $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing slopes $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$, is called the *Harder-Narasimhan filtration* of $\mathcal{S}$.

The Harder-Narasimhan filtration exists uniquely (by a Theorem of Harder and Narasimhan). A Harder-Narasimhan filtration is called strong if all the quotients $\mathcal{S}_i/\mathcal{S}_{i-1}$ are strongly semistable. A Harder-Narasimhan filtration is not strong in general, however, by a Theorem of A. Langer, there exists some Frobenius pull-back $F^{e*}(\mathcal{S})$ such that its Harder-Narasimhan filtration is strong.

Theorem 3.7. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Let*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

be a strong Harder-Narasimhan filtration. Then the torsor $T(c)$ is an affine scheme if and only if the following (inductively defined property starting with t) holds: there is an i such that $\deg(\mathcal{S}_i/\mathcal{S}_{i-1}) < 0$ and the image of c in this sheaf is $\neq 0$ (and also all Frobenius pull-backs of this class are $\neq 0$).

3.4. **Plus closure in dimension two.**

Let K be a field and let R be a normal two-dimensional standard-graded domain over K with corresponding smooth projective curve C. A homogeneous $\mathfrak{m}$-primary ideal with homogeneous ideal generators $f_1, \ldots, f_n$ and another homogeneous element f of degree m yield a cohomology class

$$c = \delta(f) \in H^1(C, \operatorname{Syz}(f_1, \ldots, f_n)(m)) .$$

Let $T(c)$ be the corresponding torsor.

We have seen in the last lecture that the affineness of this torsor over C is equivalent to the affineness of the corresponding torsor over $D(\mathfrak{m}) \subseteq \operatorname{Spec}(R)$. Now we want to understand what the property $f \in I^+$ means for c and for $T(c)$. Instead of the plus closure we will work with the graded plus closure $I^{+\mathrm{gr}}$, where $f \in I^{+\mathrm{gr}}$ holds if and only if there exists a finite graded extension $R \subseteq S$ such that $f \in IS$. The existence of such an S translates into the existence of a finite morphism

$$\varphi : C' = \operatorname{Proj}(S) \longrightarrow \operatorname{Proj}(R) = C$$

such that $\varphi^*(c) = 0$. Here we may assume that C' is also smooth. Therefore we discuss the more general question when a cohomology class $c \in H^1(C, \mathcal{S})$, where $\mathcal{S}$ is a locally free sheaf on C, can be annihilated by a finite morphism

$$C' \longrightarrow C$$

of smooth projective curves. The advantage of this more general approach is that we may work with short exact sequences (in particular, the sequences coming from the Harder-Narasimhan filtration) in order to reduce the problem to semistable bundles which do not necessarily come from an ideal situation.

Lemma 3.8. *Let C denote a smooth projective curve over an algebraically closed field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then the following conditions are equivalent.*

(1) *There exists a finite morphism*

$$\varphi : C' \longrightarrow C$$

from a smooth projective curve C' such that $\varphi^*(c) = 0$.

(2) *There exists a projective curve $Z \subseteq T$.*

Beweis. If (1) holds, then the pull-back $\varphi^*(T) = T \times_C C'$ is trivial (as a torsor), as it equals the torsor given by $\varphi^*(c) = 0$. Hence $\varphi^*(T)$ is isomorphic to a vector bundle and contains in particular a copy of C'. The image Z of this copy is a projective curve inside T.

If (2) holds, then let C' be the normalization of Z. Since Z dominates C, the resulting morphism

$$\varphi : C' \longrightarrow C$$

is finite. Since this morphism factors through T and since T annihilates the cohomology class by which it is defined, it follows that $\varphi^*(c) = 0$. □

We want to show that the cohomological criterion for (non)-affineness of a torsor along the Harder-Narasimhan filtration of the vector bundle also holds for the existence of projective curves inside the torsor, under the condition that the projective curve is defined over a finite field. This implies that

tight closure is (graded) plus closure for graded $\mathfrak{m}$-primary ideals in a two-dimensional graded domain over a finite field.

3.5. **Annihilation of cohomology classes of strongly semistable sheaves.**

We deal first with the situation of a strongly semistable sheaf $\mathcal{S}$ of degree 0. The following two results are due to Lange and Stuhler. We say that a locally free sheaf is étale trivializable if there exists a finite étale morphism $\varphi : C' \to C$ such that $\varphi^*(\mathcal{S}) \cong \mathcal{O}_{C'}^r$. Such bundles are directly related to linear representations of the étale fundamental group.

Lemma 3.9. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over C. Then $\mathcal{S}$ is étale trivializable if and only if there exists some n such that $F^{n*}\mathcal{S} \cong \mathcal{S}$.*

Theorem 3.10. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over C of degree 0. Then there exists a finite mapping*

$$\varphi : C' \longrightarrow C$$

such that $\varphi^*(\mathcal{S})$ is trivial.

Beweis. We consider the family of locally free sheaves $F^{e*}(\mathcal{S})$, $e \in \mathbb{N}$. Because these are all semistable of degree 0, and defined over the same finite field, we must have (by the existence of the moduli space for vector bundles) a repetition, i.e. $F^{e*}(\mathcal{S}) \cong F^{e'*}(\mathcal{S})$ for some $e' > e$. By Lemma 3.9 the bundle $F^{e*}(\mathcal{S})$ admits an étale trivialization $\varphi : C' \to C$. Hence the finite map $F^e \circ \varphi$ trivializes the bundle. □

Theorem 3.11. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over C of nonnegative degree and let $c \in H^1(C, \mathcal{S})$ denote a cohomology class. Then there exists a finite mapping*

$$\varphi : C' \longrightarrow C$$

such that $\varphi^*(c)$ is trivial.

Beweis. If the degree of $\mathcal{S}$ is positive, then a Frobenius pull-back $F^{e*}(\mathcal{S})$ has arbitrary large degree and is still semistable. By Serre dualtiy we get that $H^1(C, F^{e*}(\mathcal{S})) = 0$. So in this case we can annihilate the class by an iteration of the Frobenius alone.

So suppose that the degree is 0. Then there exists by Theorem 3.10 a finite morphism which trivializes the bundle. So we may assume that $\mathcal{S} \cong \mathcal{O}_C^r$. Then the cohomology class has several components $c_i \in H^1(C, \mathcal{O}_C)$ and it is enough to annihilate them separately by finite morphisms. But this is

possible by the parameter theorem of K. Smith (or directly using Frobenius and Artin-Schreier extensions). □

3.6. **The general case.**

We look now at an arbitrary locally free sheaf $\mathcal{S}$ on C, a smooth projective curve over a finite field. We want to show that the same numerical criterion (formulated in terms of the Harder-Narasimhan filtration) for non-affineness of a torsor holds also for the finite annihilation of the corresponding cohomology class (or the existence of a projective curve inside the torsor).

Theorem 3.12. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over C and let $c \in H^1(C, \mathcal{S})$ denote a cohomology class. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t$ be a strong Harder-Narasimhan filtration of $F^{e*}(\mathcal{S})$. We choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. We set $\mathcal{Q} = F^{e*}(\mathcal{S})/\mathcal{S}_i$. Then the following are equivalent.*

(1) *The class c can be annihilated by a finite morphism.*
(2) *Some Frobenius power of the image of $F^{e*}(c)$ inside $H^1(C, \mathcal{Q})$ is 0.*

Beweis. Suppose that (1) holds. Then the torsor is not affine and hence by Theorem 3.7 also (2) holds.

So suppose that (2) is true. By applying a certain power of the Frobenius we may assume that the image of the cohomology class in $\mathcal{Q}$ is 0. Hence the class stems from a cohomology class $c_i \in H^1(C, \mathcal{S}_i)$. We look at the short exact sequence

$$0 \longrightarrow \mathcal{S}_{i-1} \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_i/\mathcal{S}_{i-1} \longrightarrow 0\,,$$

where the sheaf of the right hand side has a nonnegative degree. Therefore the image of c_i in $H^1(C, \mathcal{S}_i/\mathcal{S}_{i-1})$ can be annihilated by a finite morphism due to Theorem 3.11. Hence after applying a finite morphism we may assume that c_i stems from a cohomology class $c_{i-1} \in H^1(C, \mathcal{S}_{i-1})$. Going on inductively we see c can be annihilated by a finite morphism. □

Theorem 3.13. *Let C denote a smooth projective curve over the algebraic closure of a finite field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then T is affine if and only if it does not contain any projective curve.*

Beweis. Due to Theorem 3.7 and Theorem 3.12, for both properties the same numerical criterion does hold. □

These results imply the following theorem in the setting of a twodimensional graded ring.

Theorem 3.14. *Let R be a standard-graded, two-dimensional normal domain over (the algebraic closure of) a finite field. Let I be an R_+-primary graded ideal. Then*

$$I^* = I^+ .$$

This is also true for non-primary graded ideals and also for submodules in finitely generated graded submodules. Moreover, G. Dietz has shown that one can get rid also of the graded assumption (of the ideal or module, but not of the ring).

4. Material for Tutorial Session

Material for Tutorial Session

The radical of an ideal

Now we look at the radical of the ideal I,

$$\operatorname{rad}(I) = \left\{ f \in R |\, f^k \in I \text{ for some } k \right\} .$$

The importance of the radical comes mainly from Hilbert's Nullstellensatz, saying that for algebras of finite type over an algebraically closed field there is a natural bijection between radical ideals and closed algebraic zero-sets. So geometrically one can see from an ideal only its radical. As this is quite a coarse closure operation we should expect that this corresponds to a quite coarse property of the morphism φ as well. Indeed, it is true that $f \in \operatorname{rad}(I)$ if and only if φ is surjective. This is true since the radical of an ideal is the intersection of all prime ideals in which it is contained. Hence an element f belongs to the radical if and only if for all residue class homomorphisms

$$\varphi : R \longrightarrow \kappa(\mathfrak{p})$$

where I is sent to 0, also f is sent to 0. But this means for the forcing equation that whenever the equation degenerates to 0, then also the inhomogeneous part becomes zero, and so there will always be a solution to the inhomogeneous equation.

Exercise: Define the radical of a submodule inside a module.

Integral closure of an ideal

Another closure operation is integral closure. It is defined by

$$\overline{I} = \left\{ f \in R |\, f^k + a_1 f^{k-1} + \ldots + a_{k-1} f + a_k = 0 \text{ for some } k \text{ and } a_i \in I^i \right\} .$$

This notion is important for describing the normalization of the blow up of the ideal I. Another characterization is that there exists a $z \in R$, not contained in any minimal prime ideal of R, such that $z f^n \in I^n$ holds for all n. Another equivalent property - the valuative criterion - is that for all ring homomorphisms

$$\theta : R \longrightarrow D$$

to a discrete valuation domain D (assume that R is noetherian) the containment $\theta(f) \in \theta(I)D$ holds.

The characterization of the integral closure in terms of forcing algebras requires some notions from topology. A continuous map

$$\varphi : X \longrightarrow Y$$

between topological spaces X and Y is called a *submersion*, if it is surjective and if Y carries the image topology (quotient topology) under this map. This means that a subset $W \subseteq Y$ is open if and only if its preimage $\varphi^{-1}(W)$ is open. Since the spectrum of a ring endowed with the Zarisiki topology is a topological space, this notion can be applied to the spectrum morphism of a ring homomorphism. With this notion we can state that $f \in \bar{I}$ if and only if the forcing morphism

$$\varphi : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(R)$$

is a universal submersion (universal means here that for any ring change $R \to R'$ to a noetherian ring R', the resulting homomorphism $R' \to B'$ still has this property). The relation between these two notions stems from the fact that also for universal submersions there exists a criterion in terms of discrete valuation domains: A morphism of finite type between two affine noetherian schemes is a universal submersion if and only if the base change to any discrete valuation domain yields a submersion. For a morphism

$$Z \longrightarrow \operatorname{Spec}(D)$$

(D a discrete valuation domain) to be a submersion means that above the only chain of prime ideals in $\operatorname{Spec}(D)$, namely $(0) \subset \mathfrak{m}_D$, there exists a chain of prime ideals $\mathfrak{p}' \subseteq \mathfrak{q}'$ in Z lying over this chain. This pair-lifting property holds for a universal submersion

$$\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$$

for any pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{Spec}(R)$. This property is stronger that lying over (which means surjective) but weaker than the going-down or the going-up property (in the presence of surjectivity).

If we are dealing only with algebras of finite type over the complex numbers $\mathbb{C}$, then we may also consider the corresponding complex spaces with their natural topology induced from the euclidean topology of $\mathbb{C}^n$. Then universal submersive with respect to the Zariski topology is the same as submersive in the complex topology (the target space needs to be normal).

Example 4.1. Let K be a field and consider $R = K[X]$. Since this is a principal ideal domain, the only interesting forcing algebras (if we are only interested in the local behavior around (X)) are of the form $K[X,T]/(X^nT - X^m)$. For $m \geq n$ this $K[X]$-algebra admits a section (corresponding to the fact that $X^m \in (X^n)$), and if $n \geq 1$ there exists an affine line over the maximal ideal (X). So now assume $m < n$. If $m = 0$, then we have a hyperbola mapping to an affine line, with the fiber over (X) being empty,

corresponding to the fact that 1 does not belong to the radical of (X^n) for $n \geq 1$. So assume finally $1 \leq m < n$. Then X^m belongs to the radical of (X^n), but not to its integral closure (which is the identical closure on a one-dimensional regular ring). We can write the forcing equation as $X^nT - X^m = X^m(X^{n-m}T - 1)$. So the spectrum of the forcing algebra consists of a (thickend) line over (X) and of a hyperbola. The forcing morphism is surjective, but it is not a submersion. For example, the preimage of $D(X)$ is a connected component hence open, but this single point is not open.

Example 4.2. Let K be a field and let $R = K[X, Y]$ be the polynomial ring in two variables. We consider the ideal $I = (X^2, Y)$ and the element X. This element belongs to the radical of this ideal, hence the forcing morphism

$$\operatorname{Spec}(K[X, Y, T_1, T_2]/(X^2T_1 + YT_2 + X) \longrightarrow \operatorname{Spec}(K[X, Y])$$

is surjective. We claim that it is not a submersion. For this we look at the reduction modulo Y. In $K[X, Y]/(Y) \cong K[X]$ the ideal becomes (X^2) which does not contain X. Hence by the valuative criterion for integral closure, X does not belong to the integral closure of the ideal. One can also say that the chain $V(X, Y) \subset V(Y)$ in the affine plane does not have a lift (as a chain) to the spectrum of the forcing algebra.

For the ideal $I = (X^2, Y^2)$ and the element XY the situation looks different. Let

$$\theta : K[X, Y] \longrightarrow D$$

be a ring homomorphism to a discrete valuation domain D. If X or Y is mapped to 0, then also XY is mapped to 0 and hence belongs to the extendend ideal. So assume that $\theta(X) = u\pi^r$ and $\theta(Y) = v\pi^s$, where π is a local parameter of D and u and v are units. Then $\theta(XY) = uv\pi^{r+s}$ and the exponent is at least the minimum of $2r$ and $2s$, hence $\theta(XY) \in (\pi^{2r}, \pi^{2s}) = (\theta(X^2), \theta(Y^2))D$. Hence XY belongs to the integral closure of (X^2, Y^2) and the forcing morphism

$$\operatorname{Spec}(K[X, Y, T_1, T_2]/(X^2T_1 + Y^2T_2 + XY) \longrightarrow \operatorname{Spec}(K[X, Y])$$

is a universal submersion.

Continuous closure

Suppose now that $R = \mathbb{C}[X_1, \ldots, X_k]$. Then every polynomial $f \in R$ can be considered as a continuous function

$$f : \mathbb{C}^k \longrightarrow \mathbb{C},\ (x_1, \ldots, x_k) \longmapsto f(x_1, \ldots, x_k)$$

in the complex topology. If $I = (f_1, \ldots, f_n)$ is an ideal and $f \in R$ is an element, we say that f belongs to the *continuous closure* of I, if there exist continuous functions

$$g_1, \ldots, g_n : \mathbb{C}^k \longrightarrow \mathbb{C}$$

such that

$$f = \sum_{i=1}^{n} g_i f_i$$

(identity of functions) (the same definition works for $\mathbb{C}$-algebras of finite type).

It is not at all clear at once that there may exist polynomials $f \notin I$ but inside the continuous closure of I. For $\mathbb{C}[X]$ it is easy to show that the continuous closure is (like the integral closure) just the ideal itself. We also remark that when we would only allow holomorphic functions $g_1, \ldots, g_n$ then we could not get something larger. However, with continuous functions we can for example write

$$X^2Y^2 = g_1X^3 + g_2Y^3 \,.$$

Continuous closure is always inside the integral closure and hence also inside the radical. The element XY does not belong to the continuous closure of $I = (X^2, Y^2)$, though it belongs to the integral closure of I. In terms of forcing algebras, an element f belongs to the continuous closure if and only if the complex forcing mapping

$$\varphi_{\mathbb{C}} : \mathrm{Spec}\,(B)_{\mathbb{C}} \longrightarrow \mathrm{Spec}\,(R)_{\mathbb{C}}$$

(between the corresponding complex spaces) admits a continuous section.

Possibilities:

Explain forcing algebras more carefully with examples.

Forcing algebras for principal ideals/rational functions: $fT - g$.

Forcing algebras in the module case (is closer to the linear setting).

For integral closure:

Explain universal submersions (and valuative criterion for it) more carefully (SGA I).

Give examples of

$$\varphi : \mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(R)$$

such that f is in the integral closure, but neither going up nor going down holds.

Exercise: A forcing algebra where $f \in (f_1, \ldots, f_n)$ is isomorphic to the homogeneous algebra $(f = 0)$.