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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecelea2@math.unl.edu

Speaker's Name: Alek Vainshtein

Talk Title: Cluster algebras and Poisson geometry

Date: 09/04/12 Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: cluster algebra, Poisson geometry, Poisson bracket.

Please summarize the lecture in 5 or fewer sentances: This lecture relates cluster algebras to Poisson geometry.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Alek Vainshtein – Cluster algebra and Poisson geometry
– based on Gekhtman – Shapiro – Vainshtein

① Poisson brackets
② Cluster structure reminder
③ Compatibility
④ Applications
⑤ Pre-symplectic structures

① Def A Poisson algebra $\mathcal{P}$ is an associative commutative algebra with an operation $\{\,,\,\} : \mathcal{P}\times\mathcal{P} \to \mathcal{P}$ (Poisson bracket) such that

(i) $\{x,y\} = -\{y,x\}$ skew sym.
(ii) linearity: $\{\alpha x + \beta y, z\} = \alpha\{x,z\} + \beta\{y,z\}$
(iii) Leibniz rule: $\{xy,z\} = x\{y,z\} + y\{x,z\}$
(iv) Jacobi identity: $\{x,\{y,z\}\} + \{y,\{z,x\}\} + \{z,\{x,y\}\} = 0$

Example
V = variety
$\mathcal{O}(V)$ = regular functions or rational functions

Def: Let $x_1, \ldots, x_n$ be a maximal collection of alg. indep. elements in $\mathcal{P}$. We call this collection a basis.

Def Say the Poisson bracket is quadratic in basis $\{x_1,\ldots,x_n\}$ if $\{x_i,x_j\} = \sum_{k,l} w_{ij}^{kl} x_k x_l$

Def: Say the Poisson bracket is diagonal quadratic if $\{x_i,x_j\} = w_{ij} x_i x_j$. Let $\Omega_x = (w_{ij})$
In this case we call $x_1, \ldots, x_n$ log-canonical coordinates.

② Cluster structures of geometric type.

$$\underbrace{\underbrace{x_1, \dots, x_n}_{\text{cluster var}}, \underbrace{x_{n+1}, \dots, x_{n+m}}_{\text{Stable (frozen) var}}}_{\text{extended cluster}}$$

$$\tilde{B} = n \begin{array}{cc} n & m \\ \left[\begin{array}{c|c} \text{skew symm.} & \quad\quad \end{array}\right] & \end{array}$$

$$x_i x_i' = \prod_{b_{ij}>0} x^{b_{ij}} + \prod_{b_{ij}<0} x_j^{-b_{ij}}$$

$$b_{kj}' = \begin{cases} -b_{kj} & k=i \text{ or } j=i \\ b_{kj} + \dfrac{|b_{kj}| b_{ij} + b_{ki}|b_{ij}|}{2} \end{cases}$$

③ Compatibility

Question: Find all Poisson brackets that are diagonally quadratic in all clusters. [we call them compatible with $\mathcal{C}(B)$].

Example:

$$B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\{x_1, x_2\} = \alpha x_1 x_2$$
$$\{x_1, x_3\} = \beta x_1 x_3$$
$$\{x_2, x_3\} = \gamma x_2 x_3$$
$$x_1' = \frac{x_2 + x_3}{x_1}$$

$$\{x_1', x_2\} = \left\{\frac{x_2+x_3}{x_1}, x_2\right\} = \frac{1}{x_1}\underbrace{\{x_2, x_2\}}_{0} + x_2\left\{\frac{1}{x_1}, x_2\right\} + x_3\left\{\frac{1}{x_1}, x_2\right\}$$

$$+ \frac{1}{x_1}\{x_3, x_2\} =$$

$$= -\frac{x_2}{x_1^2}\{x_1, x_2\} - \frac{x_3}{x_1^2}\{x_1, x_2\} - \gamma \frac{x_2 x_3}{x_1}$$

$$= -\frac{\alpha x_2^2}{x_1} - \frac{\alpha x_2 x_3}{x_1} - \gamma \frac{x_2 x_3}{x_1} = -\alpha \frac{x_2 + x_3}{x_1} \cdot x_2 - \gamma \frac{x_2 x_3}{x_1}$$

$\Rightarrow \gamma = 0$

Mutating in other directions $\Rightarrow \alpha = \beta = \gamma = 0 \Rightarrow$

$\Rightarrow$ there is no compatible Poisson bracket

Possible reason: $\operatorname{rank} B < n = 3$.

From now on we require $\operatorname{rank} B = n$ (B is $n \times (n+m)$)

New set of variables (<u>τ-coordinates</u>)

$$\tau_j = x_j^{\varkappa_j} \prod_{k=1}^{n+m} x_k^{b_{jk}} \qquad \varkappa_i = \begin{cases} 0, & j \le n \\ 1, \varkappa, & j > n \end{cases}$$

<u>Claim</u>: If $\operatorname{rank} B = n$, then $\tau_1, \dots, \tau_{n+m}$ are alg. indep.

<u>New transfer rule</u> $\tau_i' = \frac{1}{\tau_i}$

$$\tau_j' = \begin{cases} \tau_j (1+\tau_i)^{b_{ij}}, & b_{ij} > 0 \\ \tau_j \left(\frac{\tau_i}{1+\tau_i}\right)^{-b_{ij}}, & b_{ij} < 0 \end{cases}$$

<u>Thm</u> Let $\operatorname{rank} B = n$ s.t. $B_{[1,m,n]}^{[1,n]}$ is not decomposable. Then a Poisson bracket is compatible with $\mathscr{C}(B) \Leftrightarrow \Omega_{\tau[1,n]}^{[1,n+m]} = \lambda B$

the matrix of the Poisson bracket in basis τ

④ <u>Applications</u>

<u>Thm 1</u>: For any cluster structure of geometric type, the seed is defined uniquely by the cluster. (i.e. if after a number of mutations the variables are the same, then the matrices are also the same).

<u>Thm 2</u>: For any cluster structure of geometric type, two clusters are adjacent iff they have exactly $n-1$ common cluster variables.

<u>Thm 3</u> [If $\operatorname{rank} B = n$] then the exchange graph does not depend on coefficients.

(5) Pre-symplectic structure & cluster str. for rank $B < n$

Def A *pre-symplectic structure* is a closed 2-form ω

$$\omega = \sum \omega^{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j} \quad \text{- in any cluster basis}$$

$$\Omega^x = (\omega^{ij})$$

Thm Let $B^{[1,n]}_{[1,n]}$ have no zero rows. [and be indecomposable]. Then a pre-symplectic structure is compatible with $\mathcal{C}(B)$ iff $\Omega^{x\,[1,n+m]}_{[1,n]} = \lambda B$.

take kernels of ω $\Rightarrow$ gives a foliation of $\mathcal{X}$ = cluster manifold

presymplectic form on $\mathcal{X}$

$\downarrow$

unique symplectic form on $\mathcal{Y}$

$\mathcal{X}$ = cluster manifold

$\downarrow$ take quotient

$\mathcal{Y}$ = secondary cluster manifold