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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Craig Huneke

Talk Title: Introduction to uniformity in commutative algebra #3

Date: 09/04/12 Time: 2:00 am / (pm) (circle one)

List 6-12 key words for the talk: symbolic powers, fat points

Please summarize the lecture in 5 or fewer sentances: This lecture adresses questions and results concerning symbolic powers.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Craig Huneke – Introduction to uniformity III

$S = k[x_1, \dots, x_n]$; $I \subseteq S$ homogeneous ; $R = S/I$

$\dim_k R_i = \dfrac{e(R)}{(\dim R - 1)!} \, i^{\dim R - 1} + O(i^{\dim R - 2})$

$e(R)$ = multiplicity

If (R, m) is local, $e(R) := e(R/m \oplus m/m^2 \oplus m^2/m^3 \oplus \dots)$

Example: ① $f = \deg d \Rightarrow e(S/(f)) = d$

local: $e(S/(f)) = \operatorname{ord}(f) = \max\{n \mid f \in m^n\}$

② if $S = k[x_1, \dots, x_n]$

S/I is Cohen-Macaulay with pure resolution

$0 \to S(-d_c) \to \dots \to S(-d_2) \to S(-d_1) \to S \to S/I \to 0$

$c = \operatorname{codim} I$

$e(S/I) = \dfrac{\prod_{j=1} d_j}{c!}$

③ $X = r$ pts $\subseteq \mathbb{P}^n$; $e(R_X) = r$

④ $e(G(2,n)) =$ Catalan number

Natural question: how does $e(-)$ behave under flat maps?

Two types I. $(R, m) \to (S, n)$ local, flat

II. $R \to R_p$ localization

I Zech: Is $e(R) \le e(S)$? (Open – Hironaka)

II $R = S/(f)$, $S = R[x]$ $e(R_p) \le e(R)$

$e(R_p) = \max\{n \mid f \in \underbrace{P^n S_p \cap S}_{P^{(n)}}\} \le \max\{n \mid f \in m^n\} = e(R)$

$\iff P^{(n)} \subseteq m^n$ (Zariski – Nagata)

In general if $S = k[x_1, \dots, x_n]$, $P \in \operatorname{Spec}(S)$

Nullstellensatz: $P = \bigcap_{\substack{m \supseteq P \\ \text{maximal}}} m$; $P^{(n)} = \bigcap_{\substack{m \supseteq P \\ \text{maximal}}} m^n$

Main Question: how do I^n and $I^{(n)}$ compare.
In particular when are they the same, $\forall n$.
Here $I = P_1 \cap \dots \cap P_r$, P_i prime
$I^{(n)} = P_1^{(n)} \cap \dots \cap P_r^{(n)}$

Examples:

① If I is generated by a regular sequence (i.e. S/I is a complete intersection), then $I^n = I^{(n)}$, $\forall n \geq 1$.

② Let X be a generic $n \times n$ matrix, $\Delta = \det X$. ($n \geq 3$)
$I = I_{n-1}(X)$
$\operatorname{adj}(X) \cdot X = \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix} \Rightarrow \Delta^{n-1} = \det(\operatorname{adj} X) \in I^n \xrightarrow{\text{Ex}} \Delta \in I^{(2)} \setminus I^2$

Open problem: R a regular local ring, $P \in \operatorname{Spec}(R)$
$f^{n-1} \in P^n \overset{?}{\Longrightarrow} f \in mP$ (Hübl) /// ← in char 0
(related to Eisenbud–Mazur conjecture on evolutions: $P^{(2)} \subseteq mP$)

③ $S = k[x,y,z]$; $\dim S/P = 1$
TFAE: (i) $P^{(n)} = P^n$, $\forall n \geq 1$
(ii) $P^{(2)} = P^2$
(iii) P is locally a complete intersection

Problem: S = regular local ring (or polynomial ring)
If $P^{(\dim S)} \subseteq P^{\dim S}$ (or $\leq \dim S$) $\Rightarrow P^{(n)} = P^n$ $\forall n \geq 1$.
Two test cases where a lot is known:
I: points in $\mathbb{P}^n$, even in $\mathbb{P}^2$
II: square-free monomial ideals

Problem: R = regular local ring / $\mathbb{C}$

P = prime

Is there always a non-zero divisor of deg 1 in $\bigoplus P^n/P^{n+1}$?

Case I: pts in $\mathbb{P}^2_{\mathbb{R}}$, $X = r$ points

$I = I_X$ functions vanishing at those pts

$\operatorname{codim} I = 2 \Rightarrow I^{(2n)} \subseteq I^n \Rightarrow I^{(4)} \subseteq I^2$ (Smith's lecture)

Problem: is $I^{(3)} \subseteq I^2$? (Bocci-Harbourne: true for generic pts; True if char $k = 2$: Hochster-Huneke)

One measure of how close I^n are to $I^{(n)}$ is to compare their least degree generators.

Notation: $\alpha(I)$ = smallest degree of a generator

Since $I^n \subseteq I^{(n)} \Rightarrow \alpha(I^n) \geq \alpha(I^{(n)}) \Rightarrow \alpha(I) \geq \frac{\alpha(I^{(n)})}{n}$

$\alpha(I^n) = n\alpha(I)$

$\lim \frac{\alpha(I^{(n)})}{n}$ exists

Nagata: Let $X = r$ generic pts in $\mathbb{P}^2_{\mathbb{C}}$.

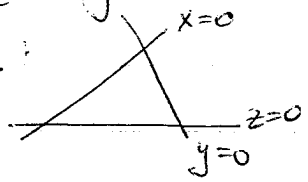
Then ① $\lim \frac{\alpha(I^{(n)})}{n} \leq \sqrt{r}$

② if r is a square, $\frac{\alpha(I^{(n)})}{n} > \sqrt{r}$

$\Rightarrow \bigoplus_{n \geq 0} I^{(n)}$ is a counterexample to Hilbert's 14th problem. (i.e. it is an infinitely generated ring of invariants).

Case II Squarefree monomials

Example:



$I_X = (x,y) \cap (x,z) \cap (y,z)$

$= (xy, xz, yz)$

$I_x^{(2)} = (x,y)^{(2)} \cap (x,z)^{(2)} \cap (y,z)^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2$

$xyz \in I^{(2)} \setminus I^2$

Generalization: I = squarefree monomial ideal $= P_1 \cap \dots \cap P_s$

P_i = subset of $(x_1, \dots, x_n)$

codim $P_i = c \Rightarrow x_1, \dots, x_n \in P_i^c \Rightarrow x_1 \cdots x_n \in P_1^c \cap \dots \cap P_s^c = I^{(c)}$

If $I^{(c)} = I^c \Rightarrow x_1, \dots, x_n \in I^c \Rightarrow$ "$\exists$ reg. seq. of c-monomials in I." (*)

If $I^{(n)} = I^n \ \forall n$, then (*) holds $\forall$ J obtained from I by setting variables equal to 0 or 1.

Are they equivalent?

(Retranslation by Gitler-Valencia-Villarreal of a "max-flow-min cut" conjecture due to Conjetos and Conforti).

Thm If I is generated by quadrics (i.e I is an edge ideal of a graph G), then TFAE

(1) $I^{(n)} = I^n \ \forall n \geq 1$.

(2) I is packed, i.e. (*) holds after setting variables $= 0$ or 1

(3) G is bipartite.

PROBLEMS FOR DISCUSSION I, INTRO WORKSHOP, MSRI

CRAIG HUNEKE

The problems below are meant to inspire discussion about uniformity. Some are very difficult, others are not so difficult. Some require only knowledge of basic algebra, while others use concepts such as Cohen-Macaulay.

1. Generators.

1. Let k be a field. Prove that every maximal ideal in $k[x_1, ..., x_n]$ is generated by n elements. In particular, every prime ideal in $k[x, y]$ is generated by at most two elements.

2. Suppose that I is a homogeneous ideal in $S = k[x_1, ..., x_n]$ generated by forms of degrees at most d, such that every variable is in the nilradical of I. Prove that I can be generated by at most the number of minimal generators of $\mathfrak{m}^d$, where $\mathfrak{m} = (x_1, ..., x_n)$. Is the same statement true if one doesn't assume that the nilradical of I contains $\mathfrak{m}$?

3. Let R be a standard graded ring over an infinite field, with homogeneous maximal ideal $\mathfrak{m}$. Say an $\mathfrak{m}$-primary maximal ideal I is $\mathfrak{m}$-full if for every general linear form ℓ, $\mathfrak{m}I : \ell = I$. Prove that if I is $\mathfrak{m}$-full and J is homogeneous and contains I, then the minimal number of generators of J is at most the minimal number of generators of I.

4. Let P be a prime ideal of a polynomial ring S such that P contains no linear forms. It is not known whether or not P is always generated by forms of degrees at most the multiplicity of S/P. Can you find examples where this estimate is sharp? What about if P is not prime?

2. Radicals

1. Let M be an n by n matrix of indeterminates over the complex numbers $\mathbb{C}$, and let I be the ideal generated by the entries of the matrix M^n. Find n equations whose nilradical equals the nilradical of I. (Hint: use linear algebra.)

2. It is an unsolved problem whether or not every non-maximal prime ideal in a polynomial ring $S = k[x_1, ..., x_n]$ can be generated up to radical by $n-1$ equations. Here is an explicit example which is not known, from Moh. Let P be the defining ideal in a polynomial ring in 3-variables of the curve $k[t^6 + t^{31}, t^8, t^{10}]$. Can you find generators of P? It is conjectured that P is generated up to radical by 2 equations. Why is this the least possible number that could generate P up to radical? If the characteristic of k is positive it is known that P is generated up to radical by 2 equations. Suppose the characteristic is 2. Find such equations.

2. Let k be the complex numbers, and let P be the defining ideal of the surface $k[t^4, t^3s, ts^3, s^4]$. Find generators for P, and find three equations which generate P up to radical. It is unknown whether or not there are 2 equations which generate P up to radical, although this is known in positive characteristic.

3. Let S be a polynomial ring, and let I be generated by forms of degrees $d_1, ..., d_s$. Suppose that f is in the nilradical of I, so that there is some N such that $f^N \in I$. Is there an effective bound? Take a guess. Find the best example you can to see that N must be large.

Date: August 28, 2012.

3. Stillman's Question

1. Let S be a polynomial ring and let I be generated by two forms. Show that the projective dimension of S/I is at most two. What well-known statement is this equivalent to?

2. Let S be a polynomial ring. It is known that if I is generated by 3 quadrics, then the projective dimension of S/I is at most 4. Find an example to see that 4 is attained, and try to prove this statement.

3. The largest known projective dimension of a quotient S/I where I is generated by 3 cubics and S is a polynomial ring is 5. Can you find such an example? (Hard.)

4. Prove the following strong form of Stillman's problem for monomial ideals: if I is generated by s monomials in a polynomial ring S, then the projective dimension of S/I is at most s.

5. What about binomial ideals? Is there a bound similar to that in questions 4?

4. Infinite Resolutions

1. If $R = S/I$ where S is a polynomial ring and I has a Grobner basis of quadrics, then R is Koszul, i.e., the residue field has a linear resolution. Prove this.

2. Suppose that $R = S/I$, where S is a polynomial ring, and I is homogeneous. If the regularity of the residue field of R has bounded regularity, show that the regularity of every finitely generated graded R-module M is also bounded.

3. Find an example of a resolution of the residue field of a standard graded algebra where the degrees of entries of matrices in a minimal resolution (after choosing bases for the free modules) are at least any fixed number N. It is a conjecture of Eisenbud-Reeves-Totaro that one can always choose bases of the free modules in the resolution of a finitely generated graded module so that the entries in all the (usually infinite) set of matrices are bounded.

4. Let R be a Cohen-Macaulay standard graded algebra which is a domain of multiplicity e. Prove that the ith total Betti number of any quotient R/I is at most e times the $(i-1)st$ total Betti number of R/I for large i. What sort of uniformity for total Betti numbers might one hope for?

5. Relations between Invariants

1. Try to imagine an effective bound conjecture relating the multiplicity of S/P, where S is a polynomial ring and P is a homogeneous prime not containing a linear form, and the regularity of S/P. Why should there be any relationship? Try the case in which P is generated by a regular sequence of forms.

2. Let S be a polynomial ring, and let I be an ideal generated by square-free monomials. Then the multiplicity of S/I is just the number of minimal primes P over I such that the dimension of S/P is maximal. Can you say anything about the regularity? For example, what if I is the edge ideal of a simple graph?

3. Is there any relationship at all between either the projective dimension (or regularity) of a quotient S/I (S a polynomial ring, I a homogeneous ideal) and the projective dimension (or regularity) of $S/\sqrt{I}$? Try to give examples or formulate a problem.

4. Do question 3 when I is generated by monomials.

6. Integral Closure

1. Let $\mathbb{C}$ be the complex numbers. Let S be the formal power series ring in n variables over $\mathbb{C}$, and let $f \in S$ with $f(0) = 0$. Set $\mathfrak{m}$ equal to the maximal ideal of S, and let f_i be the ith partial derivative of f. Prove that $f \in \overline{\mathfrak{m}I}$, where I is the ideal generated by $f_1, ..., f_n$. It is not known whether or not one can take the $\mathfrak{m}$ outside the integral closure, i.e., whether f is never a minimal generator of the integral closure of its partial derivatives. If so, this would give a positive solution to the Eisenbud-Mazur conjecture (see later problems on symbolic powers).

2. Let $f(t), g(t)$ be polynomials with coefficients in a ring R, say $f(t) = a_n t^n + ... + a_0$, and $g(t) = b_n t^n + ... + b_0$. Let c_i be the coefficient of t^i in the product fg. Prove that the ideal generated by $a_i b_j$ is integral over the ideal generated by $c_{2n}, ..., c_0$.

3. Let S be a polynomial ring in n variables, and let $g_1, ..., g_n$ be a regular sequence of forms of degree d (equivalently assume that they are forms of degree d, and the nilradical of the ideal they generator is the homogeneous maximal ideal. Prove that the integral closure of $(g_1, ..., g_n)$ is $\mathfrak{m}^d$.

7. Multiplicity

1. A famous theorem of Rees says that if R is a Noetherian local ring which is formally equidimensional (its completion is equidimensional), and I is primary to the maximal ideal $\mathfrak{m}$, then f is in the integral closure of I if and only if the multiplicity of I and the multiplicity of (I, f) are the same. Prove the easy direction of this theorem.

2. Let S be a formal power series ring in n variables, and let $f_i = x_1^{a_{i1}} + ... + x_n^{a_{in}}$. Assume that the ideal generated by the f_i is finite. Give a formula, in terms of the a_{ij}, for the length of $S/(f_1, ..., f_n)$ (which is the multiplicity of this ideal).

Exercises, Talk 3, MSRI

September, 2012

Craig Huneke

If not explicitly defined, the base ring is a polynomial ring, or power series ring, over a field, and ideals are self-radical.

1. Let P be a prime ideal generated by a regular sequence in a regular local ring (or polynomial ring). Prove that $P^{(n)} = P^n$ for all $n \geq 1$.

2. Prove that $I^{(n)} \cdot I^{(m)} \subset I^{(n+m)}$.

3. Prove that the graded algebra $T := \oplus I^{(n)}$ is Noetherian if and only if there exists an integer k such that for all n,

$$(I^{(k)})^n = I^{(kn)}.$$

4. Let S be a regular local ring and let $f \in S$ be a nonzero, nonunit element in S. Prove that the multiplicity of $R := S/(f)$ is the order of f.

5. Prove the following result of Chudnovsky, which was proved by him using transcendental methods: if I is the ideal of a set of points in the projective plane over the complex numbers, then

$$\alpha(I^{(N)}) \geq \frac{N\alpha(I)}{2},$$

where in general $\alpha(\ \ )$ denotes the least degree generator of a homogeneous ideal.

6. Let I be the ideal of at most five points in the projective plane. Prove that $I^{(3)} \subset I^2$.

7. Let Let I be an ideal of points in the projective plane over a field of characteristic 2. Prove that $I^{(3)} \subset I^2$.

8. The Eisenbud-Mazur conjecture states that if S is a power series ring over a field of characteristic 0, then for every prime ideal P,

$$P^{(2)} \subset \mathfrak{m}P.$$

Prove this if P is homogeneous.

9. Prove the Eisenbud-Mazur conjecture if for every $f \in S$ (S as in problem 8, and f not a unit), f is not a minimal generator of the integral closure of its partial derivatives.

10. Let R be a regular local ring, and let P be a prime ideal. Set $G = \oplus P^i/P^{i+1}$, the associated graded ring of P. If $f \in R$, write f^* for the leading form of f in G. Show that f^* is nilpotent in G implies that $f \in P^{(n)}$ but $f \notin P^n$ for some n.

11. Let I be a self-radical homogeneous ideal in a polynomial ring S. Prove that the limit of $\frac{\alpha(I^{(m)})}{m}$ exists as m goes to infinity.