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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Frank-Olaf Schreyer

Talk Title: Syzygies, finite length modules & random curves #2

Date: 09/05/12 Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: linkage, curve, Hartshorn-Rao module, syzygy, Betti numbers

Please summarize the lecture in 5 or fewer sentances: The lecture presents liaison theory for space curves with focus on explicit computations.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Frank Schreyer – Syzygies, finite length modules and random curves II

Proof of Petri's thm

$C \hookrightarrow \mathbb{P}^{g-1}$, $p_1, \dots, p_g \in C$ general points

$x_1, \dots, x_{g-2}, u, v$

GB for I_C

(1) $f_{ij} = x_i x_j - \sum_{r=1}^{g-2} a_{ij}^r x_r - b_{ij}$, $a_{ij}, b_{ij} \in k[u,v]$

$\binom{g-2}{2}$ quadrics, $1 \le i < j \le g-2$

(2) $g_{jk} = \alpha_j x_j^2 - \alpha_k x_k^2 +$ lower terms ($g-3$ cubics)

where $\alpha_j \in k[u,v]$ is tangent to C in p_j

(3) h = degree 4 BG element

Buchberger's test syzygies:

$x_k f_{ij} - x_j f_{ik} - \sum_{r \neq k} a_{ij}^k f_{rk} + \sum_{r \neq j} a_{ik}^r f_{rj} + S_{ijk} g_{jk} = 0$

$\Rightarrow S_{ijk} \alpha_j = a_{ij}^j$

$a_{ik}^j = S_{ijk} \alpha_k$

C irreducible $\Rightarrow$ may assume either all $S_{ijk} \neq 0$

$\Rightarrow I_C$ generated by quadrics

or all $S_{ijk} = 0 \Rightarrow f_{ij}$'s form a GB $\Rightarrow V(f_{ij}) =$ surface of degree $g-2$

u, v form a regular sequence $\Rightarrow V(f_{ij}, u, v) = V(x_i x_j, u, v)$

$\parallel$

$\{p_1, \dots, p_{g-2}\}$

2nd case: Bertini $X = V(f_{ij}\text{'s})$ is a rational normal surface scroll $\Rightarrow C \xrightarrow{3:1} \mathbb{P}^1$ or $X = \mathcal{V}_2(\mathbb{P}^2) \subseteq \mathbb{P}^5 \Rightarrow$

$\Rightarrow C \cong$ plane quintic

Rk. If C is reducible canonically embedded curve, then
$\beta_{13} \in \{0, 1, \dots, g-5, g-3\}$
Question: is $\beta_{13} = g-4$ possible?

3) Finite length modules and space curves

Def: $C \subset \mathbb{P}^3$ is a curve, if it is a CM subscheme of pure dimension 1.

$S = k[x_0, \dots, x_3]$

$C \leftarrow S/I_C \leftarrow S \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow 0$

$F_i = \bigoplus_j S(-j)^{\beta_{ij}}$

$F_3 = 0 \iff S/I_C$ is CM ($\iff C$ is ACM)

Sheafify:

$$\begin{tikzcd}
0 & \mathcal{O}_C \arrow[l] & \mathcal{O}_{\mathbb{P}^3} \arrow[l] & \bigoplus \mathcal{O}(-j)^{\beta_{ij}} \arrow[l] \arrow[dl] & \mathcal{G} \arrow[l] & 0 \arrow[l] \\
 & & \mathcal{J}_C \arrow[u] \arrow[dl] & & & \\
 & 0 & & & &
\end{tikzcd}$$

$\mathcal{G} = \ker(\mathcal{F}_1 \to \mathcal{I}_C)$ is locally free, a v. bundle

$0 = H^1_* \mathcal{G} \quad (= \bigoplus_n H^1 \mathcal{G}(n))$

$H^2_* \mathcal{G} \cong H^1_* \mathcal{J}_C = M_C$ Hartshorne–Rao module

C, $f, g \in I_C$ homogeneous regular sequence

$X = V(f,g)$, $d = \deg f$, $e = \deg g$

$X = C \cup C'$, $I_{C'} = \langle f, g \rangle : I_C$

Apply $\mathcal{E}xt^2(-, \omega_{\mathbb{P}})$: $0 \to \mathcal{J}_{C|X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$

$0 \leftarrow \mathcal{E}xt^2(\mathcal{J}_{C|X}, \omega_{\mathbb{P}}) \leftarrow \omega_X \leftarrow \omega_C \leftarrow 0$

$$\begin{tikzcd}
\mathcal{O}_C & \mathcal{O} \arrow[l] & \bigoplus \mathcal{O}(-j)^{\beta_{ij}} \arrow[l] & \mathcal{G} \arrow[l] & 0 \arrow[l] \\
\mathcal{O}_X \arrow[u] & \mathcal{O} \arrow[l] \arrow[u, equal] & \mathcal{O}(-d) \oplus \mathcal{O}(-e) \arrow[l] \arrow[u] & \mathcal{O}(d-e) \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

Apply $\mathcal{H}om(-, \mathcal{O}(-d-e))$ and use $\omega_X \cong \mathcal{O}_X(d+e-4)$

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus \mathcal{O}(j-d-e)^{\beta_{1j}} \arrow[r] \arrow[d] & \mathcal{G}^* \arrow[r] \arrow[d] & \omega_C(d+e-4) \arrow[d] \\
0 \arrow[r] & \mathcal{O}(-e) \oplus \mathcal{O}(-d) \arrow[r] \arrow[ur, dash, "\oplus"] & \mathcal{O} \arrow[r] \arrow[dr] & \mathcal{O}_X \\
& & & \mathcal{O}_{C'}
\end{tikzcd}$$

$\mathcal{E}xt^2$

$$M_{C'} = \bigoplus H^1 \mathcal{J}_{C'}(n) = \bigoplus H^1 \mathcal{G}^*(n-d-e)$$
$$= \bigoplus H^2 \mathcal{G}(-4+d+e-n)^* = \mathrm{Hom}_R(M_C, k)(d+e-4)$$

Serre Duality

<u>Thm</u> (Rao '78)
The even liaison classes of curves in $\mathbb{P}^3$ are in bijection with finite length graded modules up to shift

Starting with C we get

$$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O} \leftarrow \begin{matrix} \mathcal{F} \\ \oplus \\ \mathcal{L}_1 \end{matrix} \leftarrow \mathcal{L}_2 \leftarrow 0$$

where $\mathcal{L}_i$ are direct sum of line bundles
$\mathcal{F}$ has no line bundle summand
and $H^1_* \mathcal{F} = H^1_* \mathcal{J}_C = M_C$

$$0 \leftarrow M_C \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow 0$$

with $F_1 \leftarrow N \leftarrow F_2$.

$\mathcal{F} = \tilde{N}$ to get C choose $\varphi \in \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1 \oplus \mathcal{F})$
Set $\mathcal{J}_C = \mathrm{coker}\, \varphi$
To get an idea how to choose M and the $\mathcal{L}_1, \mathcal{L}_2$ use

$$H_M(t) = \sum \dim M_n t^n = \frac{\sum_j \left(\sum_i (-1)^i \beta_{ij}\right) t^j}{(1-t)^4}$$

<u>Example</u> $d = 11$, $g = 10$,

$C \subset \mathbb{P}^3$ linearly normal (i.e exactly 4 sections)

n	$h^1(\mathcal{I}_C)$	$h^0\mathcal{O}_C(n)$		$h^0\mathcal{O}_{\mathbb{P}^3}(n)$
0		1		1
1		4		4
2	3	$22+1-10=13$		10
	4	24		20
		35	$\overset{\vee}{=}$	35
		46		56

$$(3t^2+4t^3)(1-t^4) = 3t^2 - 8t^3 + 2t^4 + 12t^5 - 13t^6 + 4t^7$$

Expected syzygy table:

$\beta(M)$:

3	8	2		
		12	13	4

$\beta(H^0_*\mathcal{O}_C)$

1			
	3	8	2
			2

$\beta(S_C)$

1			
4	10	13	4