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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu **Email/Phone:** asecelean2@math.unl.edu

Speaker's Name: Frank-Olaf Schreyer

Talk Title: Syzygies, finite length modules & random curves #3

Date: 09/06/12 **Time:** 9:00 am / pm (circle one)

List 6-12 key words for the talk: moduli space, linkage, Betti numbers, curve, Hartshorn-Rao module

Please summarize the lecture in 5 or fewer sentances: Unirationality proofs of moduli spaces of space curves how computer aided.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

F. Schreyer — Syzygies, finite length modules and random curves III.

4) Random curves

Def A variety M is unirational if there exists a dominant rational map $\mathbb{A}^N \dashrightarrow M$.

If a moduli space is unirational, then in principle we can construct a family

$$\begin{array}{l} \mathcal{C} \\ \downarrow \\ \mathbb{A}^N \setminus B \end{array}$$

such that the fibers $\mathcal{C}_t$ fill a Zariski open subset of $\mathcal{C} \subset M$.

Along a unirationality proof we should be able to compute such a family theoretically. In practice, this is out of reach in most cases for computer algebra systems.

However, if we choose specific values $t \in K^N$, then the computation of a single fiber will often be possible.

If $K = \mathbb{F}$ is a finite field, $t \in \mathbb{F}^N$ at random one gets random points in $M(\mathbb{F})$

In this lecture, it will be shown how this works for $\mathcal{M}_g$ = the moduli space of curves

Positive results

$\mathcal{M}_g$ is unirational if

$g \leq 10$ Severi (plane nodal curves)

$g = 11, \dots, 13$ Chang - Ran, Severi '80

$g = 14$ Verra 200x

Negative results

Thm (Harris Mumford, Eisenbud Harris, Farkas)

$\mathcal{M}_g$ is of general type for $g \geq 24$ or $g = 22$.

Problem: Close the gap between 14 and 22.

To start, we have to choose which model of C we would like to construct.

Brill - Noether theory: let C be a smooth curve

$W^r_d(C) = \{ L \in \mathrm{Pic}^d(C) \mid h^0(L) \geq r+1 \}$

<u>Thm</u> (Brill-Noether, Griffith-Harris, Fulton-Lazarsfeld, Gieseker, Lazarsfeld)

1) W^r_d has dim. $\geq \rho := g-(r+1)(g+r-d)$ in every component

2) if $\rho \geq 0$, $W^r_d \neq \emptyset$ and if $\rho > 0$, W^r_d is connected

3) If C is a general curve, then

$W^r_d \setminus W^{r+1}_d$ is smooth of dimension ρ at every point

Indeed, for $L \in W^r_d \setminus W^{r+1}_d$ we have

$$T_L W^r_d = \mathrm{Im}\big(\underset{\text{Petri map}}{\mu_L}: H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \longrightarrow H^0(\omega_C)\big)^{\perp} \subset T_L \mathrm{Pic}^d = H^1 \mathcal{O}_C$$

For C general, all μ_L are injective.

<u>Ex</u> <u>Unirationality of $\mathcal{M}_{12}$</u>.

<u>Step 1</u>: Choose g, d, r such that $W^r_d \neq \emptyset$ for a general curve

<u>Ex</u> $g=12$, $d=13$, $r=3 \Rightarrow C \subset \mathbb{P}^3$

$$\underset{\overset{\shortparallel}{4}}{h^0 \mathcal{O}_C(1)} - \underset{\overset{\shortparallel}{2}}{h^1 \mathcal{O}_C(1)} = 13-1-12$$

$\rho \geq 0 \Rightarrow W^r_d \neq \emptyset \Rightarrow$ every space curve has such a model.

<u>Step 2</u> Compute the Hilbert function of the Hartshorne-Rao module $M = H^1_* \mathcal{I}_C$ under the assumption that $H^0 \mathcal{O}_C(n) \longleftarrow H^0 \mathcal{O}_{\mathbb{P}^3}(n)$ is maximal rank, $\forall n$.

<u>Ex</u> $H_M(t) = 5t^2 + 8t^3 + 4t^4$

<u>Step 3</u> Determine the expected Betti numbers of M and

$H^1_*(\mathcal{L}_C)$

Ex $\beta(M) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 2 & 5 & 12 & 4 & & \\ & & & 4 & & \\ & & & 9 & 16 & 5 \end{array}$ $\quad \beta(H^0_*(\mathcal{O}_C)) = \begin{array}{|ccc} 1 & & \\ - & & \\ 5 & 12 & 4 \\ & 0 & 2 \end{array}$

$2 \quad 4+4=12$

dual is $\beta(H^0_* \omega_C)$

$h^0 \omega_C(-1)$

means that the Petri map is injective i.e. there

$H^0\mathcal{O}_C(1) \otimes H^0\omega_C(-1) \longrightarrow H^0\omega_C$ are no linear relations

$\mathcal{F} = \ker\left(\mathcal{O}(-3)^{12} \xrightarrow{d^0(M)} \mathcal{O}(-2)^5 \right)$ vector bundle of rank 7

Hope $\operatorname{coker}\left(\mathcal{F} \xleftarrow{\varphi} \begin{array}{c} \mathcal{O}(-4)^4 \\ \oplus \\ \mathcal{O}(-5)^2 \end{array} \right) = \mathcal{I}_C$ is a smooth curve

$G(2,4)$

there is a choice of $\mathcal{O}(-5)^2 \subset \mathcal{O}(-5)^4$ from Betti table.

Step 4 Construct a module M with desired syzygies (difficult part)

Step 5 Check that $\operatorname{coker}\varphi$ defines a curve of right degree and genus and it is smooth.

Ex (Step 4)

$$\begin{array}{ccc} 5 & 12 & 4 \\ & & 4 \\ & & 9 \quad 16 \quad 6 \end{array}$$

Idea: pick a random 5×12 matrix of lin. forms

this produces a module w/ HF $5, 8, 0 \dots \leftarrow$ bad.

Instead pick a linear map $12 \leftarrow 4$ i.e.

$$S \longrightarrow S(1)^{12} \xleftarrow{x^t} S(2)^4$$

$$\underset{\text{expected}}{8} \qquad 4 \cdot 12 \longrightarrow 40$$

Next step choose $\underbrace{S^5 \hookrightarrow S^8 \longrightarrow}_{\beta} S^{12}(1)$

Set $M = \operatorname{coker}\left(S^{12}(-3) \xrightarrow{\beta^t} S^5(-2)\right)$

$$\begin{array}{ccccccc} M \leftarrow S(-2)^5 & \xleftarrow{\beta^t} & S(-3)^{12} & \longleftarrow & S^4(-4) & = & S^4(-4) \\ & & & & \oplus & & \oplus \\ & & \mathcal{F} & & S^4(-5) & \hookleftarrow & S^2(-5) \\ & & & & \oplus & & \\ & & & & S^9(-6) & & \end{array}$$

(Step 5)

Because μ_L is injective $\Rightarrow$ $\mathcal{A}_{d,g}/PGL(r+1) \xrightarrow{\text{generically say}} W^r_d(\mathcal{M}_g)$

$$\mathbb{A}^N \longrightarrow \mathcal{M}_g$$

<u>An example that I cannot do</u>

$g = 16 = 4 \cdot 4, \quad d = 15$

$H_M(t) = 5t^2 + 10t^3 + 10t^4 + 4t^5$

Expected syzygies: 5 10 —

(this is known to — 4

be correct for a 9 6

general $C \in \mathcal{M}_g$) 6 4

$\beta(H^0_* \mathcal{O}_C)$

1		
—		
5	10	
	4	

$\beta(S/J_C) =$

1		
9	6	
	6	4

$\Gamma = 20$ determinantal points in $\mathbb{P}^3$ (V(minors of 4×6 lin. matrix))

6 9

4 6

6 5 — determinantal curve E, $d = 16$, $g = 26$

$\Gamma \subset E \quad H^0 \mathcal{O}_E(\Gamma) = 1.$

Γ, E should be constructed simultaneously