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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecelanu2@math.unl.edu

Speaker's Name: Michel Van den Bergh

Talk Title: Non-commutative resolutions

Date: 09/06/12 Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: non-commutative resolution, Gorenstein, crepant, endomorphism ring

Please summarize the lecture in 5 or fewer sentances: Non-commutative resolutions generalize the algebraic geometric concept of crepant resolutions of singularities. Existence and non-existence results are presented.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Noncommutative Resolutions #1 9/6/12

Michel Van den Bergh

http://hardy.uhasselt.be/msri_2012.pdf

Introduction

Let X be an algebraic variety, (singular)

$\pi : Y \longrightarrow X$ resolution, Y smooth
proper birational ?

If X a surface, $\exists!$ minimal, but in general this is not the case

Example: $X - \{xy - zt\}$ blowing up (x, z) or (x, t) give 2 different resolutions

Definition A resolution π is crepant if
(1) X is Gorenstein (X is CM, ω_X invertible)
(2) $\pi^* \omega_X = \omega_Y$

The two resolutions in the example are crepant.

General Principle
Different crepant resolutions have similar properties.

E.g. same Hodge #'s (Kontsevich).

Conjecture (Bondal Orlov)
All crepant resolutions are derived equivalent (relative to X).

$$\begin{tikzcd} Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} D^b(\mathrm{coh}\, Y_1) \arrow[rr, "\cong"] \arrow[dr, "L\pi_*"'] & & D^b(\mathrm{coh}\, Y_2) \\ & D^b(\mathrm{Per}(X)) \arrow[ur, "L\pi^*"'] & \end{tikzcd}$$

~~Define~~

B.O. Conjecture has been proved by Bridgeland in dimension 3.

Strategy: analyze flops.

flop: $Y_1 \overset{\text{flop}}{\dashrightarrow} Y_2$, $\pi_1: Y_1 \to X \leftarrow Y_2 : \pi_2$ $\quad \mathrm{Div}(Y_1) \simeq \mathrm{Div}(Y_2)$

rel. ample $L \rightsquigarrow$ anti rel ample.

Proof showed: $\exists$ sheaf of non-comm. algebras $\mathcal{A}$ such that.

$$D^b(\mathrm{coh}\, Y_1) \cong D^b(\mathrm{coh}\, \mathcal{A}) \simeq D^b(\mathrm{coh}\, Y_2)$$

Example (Atiyah Flop)

$X = \mathrm{Spec}\, R \qquad R = k[x,y,z,t]/(xy-zt)$

$D^b(\mathrm{coh}\, Y_1) \cong D^b(\mathrm{mod}\, A) \cong D^b(\mathrm{mod}\, Y_1)$

$A = \begin{pmatrix} R & I \\ I^* & R \end{pmatrix} \quad I = (Y, Z) \quad I^* \cong X^{-1}(x,t)$

non commutative crepant resolution.

Definition R normal, Gorenstein. A non-commutative crepant resolution (NCCR) is an R-algebra A such that

(1) $\exists$ reflexive R-module M s.t.

$A \cong \mathrm{End}_R M$

birationality

(2) A is MCM as an R-module $\quad$ crepant condition

← max'l CM

(3) $\mathrm{gldim}\, A < \infty$ $\quad$ smoothness analogue

Comments

1. Sometimes we call M the NCCR instead of A

2. In all known cases, we can take M to be MCM as well.

$\boxed{\text{Lemma}}$

If A is an NCCR and $P \in \mathrm{Spec}\, R$ then $\mathrm{gldim}\, A_P = \mathrm{kdim}\, R_P$

Comment:

What is a non-commutative (not necessarily crepant) resolution?

Very Weak Proposal (Dao, Iyama, Takahashi, Vial)

R a commutative Noetherian ring. A NCR (non comm. resolution) is an R-algebra ~~and~~ $\mathrm{End}_R M$ of finite gl dim s.t. M is faithful, i.e. $R \hookrightarrow \mathrm{End}_R M$. $\quad M \in \mathrm{mod}\, R$

~~Example (McKay setting)~~

~~Suppose V a finite dimensional vectorspace, $\dim V = d$,~~

Example (McKay Setting)
Suppose V a finite dimensional vector space, $\dim V = d$. $R = S^G$, $S := \mathrm{Sym}\, V$,
$G \subseteq GL(V)$ finite $\to$ no element fixes a hyperplane point wise (no pseudo-reflexions)
S/R unramified in codim 1
$S \# G \xrightarrow{\cong} \mathrm{End}_R(S)$
gldim d NCR with $M = {}_RS \in MCM(R)$.

If $G \subseteq Sl(V)$. $\Rightarrow R$ is Gorenstein
$\Rightarrow A$ is a NCCR.

Relation to Cluster Categories (Iyama)

R (complete) local Gorenstein, isolated singularity. $\mathrm{Kdim}\, R = d$

$\boxed{\text{Theorem}}$ (Iyama)
(i) If $M \in MCM(R)$ such that $\mathrm{Hom}_R(M,M) \in MCM(R)$, then $\mathrm{Ext}_R^{1,\dots,d-2}(M,M) = 0$.

(ii) If $M \in MCM(R)$ is a NCCR, then for $N \in MCM(R)$, $\mathrm{Ext}_R^{1,\dots,d-2}(M,N) = 0 \Rightarrow N \in \mathrm{add}(M)$.
additive category: sums of summands

$\boxed{\text{Theorem}}$ (Auslander)
The category $\mathcal{C} = \underline{MCM}(R)$ is $d-1$ Calabi Yau

Hence, Iyama's result implies that M is a $d-2$ cluster-tilting object in $\mathcal{C}$

Endomorphism Rings of finite Global Dimension

dim 0 A fin. dim. over k, not nec. commutative.

$\boxed{\text{Theorem}}$ (Iyama)
$\exists\, N \in \mathrm{mod}\, A$ such that gl dim $\mathrm{End}_A M < \infty$
$M = A \oplus A^* \oplus N$
$A^* = DA = \mathrm{Hom}_k(A, k)$

dim 1 Assume hereon that R is complete, local, and Cohen-Macaulay

$\boxed{\text{Theorem}}$ (Leuschke)
$k \dim R = 1$. $\exists\, M \in MCM(R)$ such that gl dim $\mathrm{End}_R(M) < \infty$

dim ≥ 2
$\exists$ Strong restrictions for the existence of a NCCR.

$\boxed{\text{Theorem}}$ (Stafford - Van den Bergh)
If R has a NCCR, then it has rational singularities.

It is not known if some version of this holds for NCRs.

$\exists$ partial results (DTTV) in the graded case

There are other strong restrictions in the hypersurface case. (DAO)