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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Secelean **Email/Phone:** asecelean2@math.uiuc.edu

Speaker's Name: Dylan Thurston

Talk Title: Cluster Algebra and Triangulated Surfaces

Date: 09/06/12 **Time:** 2:00 **am / pm (circle one)**

List 6-12 key words for the talk: cluster algebra, triangulation, surface

Please summarize the lecture in 5 or fewer sentances: This lecture discusses connections between cluster algebras and triangulations of surfaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Cluster Algebras and Triangulated Surfaces

Dylan Thurston 9/6/12

Surface Cluster Algebras

I. Ptolemy relation + other basics
II. Coefficients and shear coordinates
III. Skein relations, positivity and canonical bases

Given a punctured surface Σ,

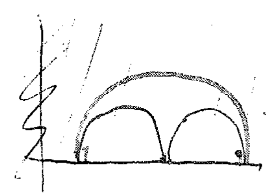
$PSL_2(\mathbb{R})$ representations $\geq$ Teichmüller space
holonomy parabolic around punctures $=$ space of hyperbolic metrics on Σ w/ cusps at puncture
positive part discrete faithful reps $=$ space of conformal structures on Σ

Gives good examples of cluster algebras
- mutationally finite
- all but finitely many mutationally finite cluster algebras

(Felikson–Shapiro–Tumarkin)

Start simpler: hyperbolic ideal polygons

Poincaré half-plane $\quad ds^2 = \dfrac{dx^2+dy^2}{y^2}$

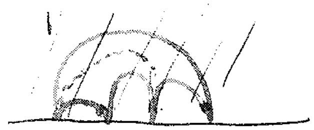


Triangles have max area as vertices approach real axis.

ideal triangle: triangle with all vertices on $\mathbb{RP}^1$
area $= \pi$

$\mathrm{Isom}(\mathbb{H}^2) = PSL_2(\mathbb{R})$
can take any 3 points in $\mathbb{RP}^1$ to each other.

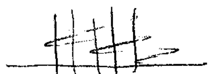
ideal quadrilaterals



How to parametrize

Decorated polygons (Penner)

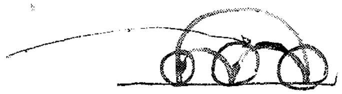
Pick a horocycle around each ideal point
(a set of points at equal distance to ideal point)



~~a circle tangent to ∂ in Poincaré model~~

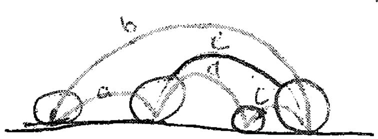
- tangent circle to ∂ in Poincaré model
- meets geodesics to ideal point at right angles

$\ell(e)$: distance between two horocycles



$\lambda(e) = e^{\ell(e)/2}$ "λ-length"

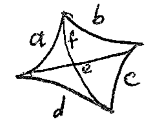
Invariant of quadrilateral:



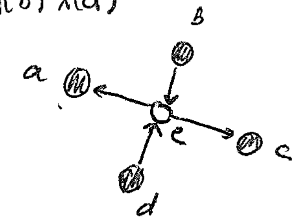
$\lambda(a), \dots, \lambda(d), \lambda(e)$

$\dfrac{\lambda(a)\,\lambda(c)}{\lambda(b)\,\lambda(d)}$ = cross ratio of 4 ideal points.

$\boxed{\text{Lemma}}$

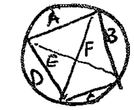


In a decorated quadrilateral, we have the relation
$\lambda(e)\lambda(f) = \lambda(a)\lambda(c) + \lambda(b)\lambda(d)$



This is called the Ptolemy relation.

Similar relation for cyclic quadrilateral in $\mathbb{E}^2$



Proof (idea)
Translate everything to linear algebra

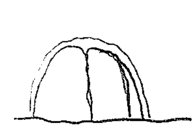
ideal point $\longleftrightarrow$ pt in $\mathbb{RP}^1$ $\longleftrightarrow$ line in $\mathbb{R}^2$ &

ideal pt + horocycle $\longleftrightarrow$ vector in $\mathbb{R}^2$ (up to ± 1) $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

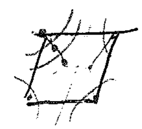
λ length $\longleftrightarrow$ determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Ptolemy relation $\longleftrightarrow$ Plücker relation in $Gr(2,4)$

Related Proof
Degenerate hyperbolic metric
ideal triangles, quadrilaterals $\to$ trees.

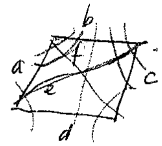


dually



polygon with simple (nonintersecting) curves w endpoints on boundary up to isotopy

Ptolemy relation, tropical version



$i(x) = \#$ of intersections w/ simple curves

Then

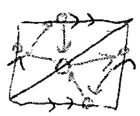
$i(e) + i(f) = \max\left(i(a) + i(c),\ i(b) + i(d) \right)$

$4 + 4 = \max \quad 3 + 3 \quad 5 + 3$

We don't actually need any of this geometry to study the associated cluster algebras.

<u>Marked surfaces</u>: surfaces, (not too small.) possibly with boundary, with some marked points on the interior or boundary, so that there is at least one marked point on each component, and at least one on each boundary component.

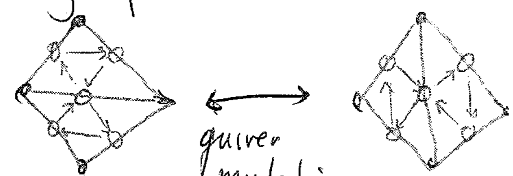
Pick a <u>triangulation</u> with vertices at marked points



(a maximal collection of disjoint arcs with endpoints at punctures)

Create a quiver for the triangulation: with one vertex for each edge of the triangulation and a frozen variable for each boundary arc. (counterclockwise arrows.)

Any two triangulations of Σ are related by quadrilateral flips.

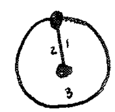


(at least generically, if all edges distinct.

$\boxed{\text{Theorem}}$

Every (sufficiently large) marked surface gives a cluster algebra, initial seed given by any triangulation (without self-folded triangles),

self-folded:



positive points of geom. cluster algebra parametrize decorated Teichmüller space of Σ.