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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: asecelean2@math.unl.edu

Speaker's Name: Michel Van den Bergh

Talk Title: Non-commutative resolutions

Date: 09/07/12 Time: 09:00 am / pm (circle one)

List 6-12 key words for the talk: non-commutative resolution, Gorenstein, crepant, endomorphism ring

Please summarize the lecture in 5 or fewer sentances: Non-commutative resolutions generalize the algebraic geometric concept of crepant resolutions of singularities. Existence and non-existence results are presented.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - Computer Presentations: Obtain a copy of their presentation
    - Overhead: Obtain a copy or use the originals and scan them
    - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Noncommutative Resolutions #2   9/7/12

Michel van den Bergh

last time: NCCR $\quad \mathrm{End}_R(M) \longrightarrow \begin{cases} \xrightarrow{} \text{reflexive} \\ \text{MCM} \\ \text{gl dim} < \infty \end{cases}$

NCR $\longleftarrow$

$k = \bar{k}$, char $k = 0$. R complete local normal

(dim 2) $\mathrm{MCM}(R) = \mathrm{ref}(R) \subset \rightarrow$ ref R modules.

$\boxed{\text{Theorem}}$ (DITV) TFAE: (Kdim 2)

(i) R has a NCR

(ii) R has rational singularities

Restrict to R rational ring

$\boxed{\text{Theorem}}$ (Artin)

If R is Gorenstein then $R \cong k[[x,y]]^G$ $G \subseteq SL_2$

$\boxed{\text{Theorem}}$ (Auslander)

If $|\,\mathrm{Indec}\,\mathrm{MCM}(R)/\cong| < \infty$ $\quad$ finite # of indecomposables $\quad R = k[[x,y]]^G$,

$G \subseteq GL_2(k)$.

In general (i.e. R not necessarily finite representation type) $\exists$ canonical NCR.

(Artin-Kercher, Wunram, Iyama-Wemyss)

Definition $M \in \mathrm{MCM}(R)$ is <u>special</u> if

$\mathrm{Ext}^1_R(M,R) = 0$

(vacuous in Gorenstein case)

$\boxed{\text{Theorem}}$ (Wunram)

The indecomposable special $\mathrm{MCM} \ncong R$ are in 1-1 correspondence with the irreducible comps of the exc. fiber in a minimal resolution (in particular the # is finite).

$\boxed{\text{Theorem}}$ (IW)

Let $M = \bigoplus M_i$ indecomposable, special

$\mathrm{gl\,dim}\,\mathrm{End}_R(M) = \begin{cases} 2 & \text{Gorenstein case} \\ 3 & \text{else} \end{cases}$

"reconstruction algebra"

Remark $G \subseteq GL_2(k)$, $\not\subseteq SL_2(k)$

$\mathrm{End}_R M$	$\ncong k[[x,y]] \# G$
gl dim 3	gl dim 2

dim 3 Restrict to R Gorenstein, Kdim 3, rational singularities.

<u>Definition</u> R has (Gorenstein) <u>terminal singularities</u> if for a (any) resolution $Y \longrightarrow X = \operatorname{Spec} R$ $\operatorname{Supp}(\omega_Y / \pi^* \omega_X)$ contains all divisors in the ex. fiber.

$\boxed{\text{Theorem}}$ (vdB)
R has an NCCR $\Rightarrow$ Spec R has a crepant resolution

$\Leftarrow$ (if R has terminal singularities)

<u>Conjecture</u> should be $\Leftrightarrow$

(Partial results by Toda, Uehara)

<u>Remark</u> The $\Rightarrow$ arrow constructs the crepant resolution as a moduli space.
<u>Idea</u>: comes from the three-dimensional McKay correspondence (Bridgeland, King, Reid)

$\boxed{\text{Theorem}}$ (Iyama-Reiten, IW)
All crepant resolutions of R are derived equivalent (non commutative & commutative)

<u>Remark</u> We conjecture that this is true in higher dimension as well
"generalized B.O. conjecture".

dim 4 Implications fail (dramatically

Ex

- $k[[x,y,z,t]]^G$ $\quad G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

factorial, terminal $\Rightarrow$ No crepant resolution.

$\exists$ NCCR (quotient singularity)

- (Dao) $k[[x,y,z,t,u]]/(f)$
isolated sing., no NCCR.

- (Lin) $f = x^5 + y^4 + z^4 + t^4 + u^4$
admits a crepant resolution.

Construction of NCCRs

- skew group ring
- tilting bundle on commutative resolution.

Typical Situation

$Y \xrightarrow{\pi} X = \operatorname{Spec} R$
↳ commutative, Noetherian, Gorenstein

$\operatorname{codim} \operatorname{Ex}(\pi) \geq 2$
↳ ex. locus

Assume $\exists$ vector bundle Z on Y such that

- $D^b(\operatorname{coh} Y) = \langle Z \rangle$ $\;(= \operatorname{thick}(Z))$
- $\operatorname{Ext}^i_{\mathcal{O}_Y}(Z, Z) = 0$ for $i > 0$

Then, $M = \Gamma(Y, Z)$ is reflexive, &
$A = \operatorname{End}_R(M)$ is a NCCR.
↳ $D^b(\operatorname{mod} A) \cong D^b(\operatorname{mod} Y)$

Construction of Z?

$\boxed{\text{Theorem}}$ (VdB)
Assume in addition that
(1) π is projective
(2) dim fibers $\pi \leq 1$
Then, Y has a tilting bundle.

In particular, this applies to 3-dimensional terminal singularities.

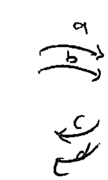
Some known cases

- cones over Del Pezzo surfaces
- determinantal varieties (Buchweitz, Leuschke, VdB)
- k^n / T, T 1-dimensional torus acting linearly (Kaczmarczyk?, VdB)
- 3-dimensional toric Gorenstein singularities (Broomhead)
- Singularities with symplectic resolutions (Kaledin)

→ $\boxed{\text{Theorem}}$ VdB (from next pg.)
$A = \widehat{J(Q,W)}$ $\quad W \in kQ/[kQ, kQ]$
$J(Q,W) = kQ / \left(\frac{\partial w}{\partial a} \right)_{a \in Q_1}$

On paths $\frac{\partial p}{\partial a} = \sum_{ras} sr$, extend linearly

Example $k[[x,y,z,t]]/(xy - zt)$

Q  $W = acbd - adcb$

Equations of NCRs

Principle: often, N(C)CRs have nice "equations"
(Quiver with relations)

dim 2 $G \subseteq SL_2(k)$ $\quad S = k[[x,y]]$ $\quad R = S^G$
$S\#G \cong \mathrm{End}_R(S)$ $\quad$ NCCR

$\boxed{\text{Theorem}}$ (Auslander-Reiten)
$S\#G$ Morita? complete and preprojective algebra of an extended Dynkin Quiver.

$G = \langle G \rangle$, $G = \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix}$ $\quad \zeta^n = 1$

a_i, a_i^*, a_{i+1}^*, a_{i+1} $\quad a_{i+1}^* a_{i+1} = a_i a_i^*$
or $\left(\sum_i [a_i^*, a_i] = 0 \right)$

$G \subseteq GL_2$, $\exists$ nice arrows (Reiten, VdB)

dim 3 R Gorenstein, complete, local, Kdim 3
M NCCR of R.

Fact $A = \mathrm{End}_R(M)$ is (complete) Ginzberg 3-CY.
(i.e. $A \in \mathrm{per}(A^e)$), $R\mathrm{Hom}_{A^e}(A, A^e) \cong A[3]$

$\boxed{\text{Theorem}}$ (VdB)
$\cancel{A = J(Q,W)}$, $\cancel{W \in kQ/[kQ,kQ]}$,