CCSS Mathematical Practices in K-5

To emphasize the Mathematical Practices, the CCSS gives them their own distinct section, but they are not to be thought of as a separate skill set to be handled in special lessons or supplements. The intent is that these *essential mathematical habits of mind and action* pervade the curriculum and pedagogy of mathematics, K–12, in age-appropriate ways. This document interprets and illustrates each of the eight Mathematical Practices as they might be exemplified in grades K–5.

Each section begins with the title of a Mathematical Practice, as given in the CCSS, followed by part or all of the CCSS description of that standard. Omissions are made only when the original text is an elementary school example that needs no further comment or is explicitly about content that goes beyond elementary school. All omissions are indicated with ellipses (…).

1. Make sense of problems and persevere in solving them.

> Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. … —CCSS

The problems we encounter in the "real world"—our work life, family life, and personal health—don't ask us what chapter we've just studied and don't tell us which parts of our prior knowledge to recall and use. In fact, they rarely even tell us exactly what question we need to answer, and they almost never tell us where to begin. They just happen. To survive and succeed, we must figure out the right question to be asking, what relevant experience we have, what additional information we might need, and where to start. And we must have enough stamina to continue even when progress is hard, but enough flexibility to try alternative approaches when progress seems too hard.

The same applies to the real life problems of children, problems like learning to talk, ride a bike, play a sport, handle bumps in the road with friends, and so on. What makes a problem "real" is not the context. A good puzzle is not only more part of a child's "real world" than, say, figuring out how much paint is needed for a wall, but a better model of the nature of the thinking that goes with "real" problems: the first task in a crossword puzzle or Sudoku or KenKen® is to figure out where to start. A satisfying puzzle is one that you don't know how to solve at first, but can figure out. And state tests present problems that are deliberately designed to be different, to require students to "start by explaining to themselves the meaning of a problem and looking for entry points to its solution."

Mathematical Practice #1 asks students to develop that "puzzler's disposition" in the context of mathematics. Teaching can certainly include focused instruction, but students must also get a chance to tackle problems that they have *not* been taught explicitly how to solve, as long as they have adequate background to *figure out* how to make progress. Young children need to build their own toolkit for solving problems, and need opportunities and encouragement to get a handle on hard problems by thinking about similar but simpler problems, perhaps using simpler numbers or a simpler situation.

One way to help students make sense of all of the mathematics they learn is to put experience before formality throughout, letting students explore problems and derive methods from the exploration. For example, students learn the logic of multiplication and division—the distributive property that makes possible the algorithms we use—before the algorithms. The algorithms for each operation become, in effect, capstones rather than foundations.

Another way is to provide, somewhat regularly, problems that ask only for the analysis and not for a numeric "answer." You can develop such problems by modifying standard word problems. For example, consider this standard problem:

> Eva had 36 green pepper seedlings and 24 tomato seedlings. She planted 48 of them. How many more does she have to plant?

You might leave off some numbers and ask children how they'd solve the problem if the numbers were known. For example:

> Eva started with 36 green pepper seedlings and some tomato seedlings. She planted 48 of them. If you knew how many tomato seedlings she started with, how could you figure out how many seedlings she still has to plant? (I'd add up all the seedlings and subtract 48.)

Or, you might keep the original numbers but drop off the question and ask what can be figured out from that information, or what questions can be answered.

> Eva had 36 green pepper seedlings and 24 tomato seedlings. She planted 48 of them. (I could ask "how many seedlings did she start with?" and I could figure out that she started with 60. I could ask how many she didn't plant, and that would be 12. I could ask what is the smallest number of tomato seedlings she planted! She had to have planted at least 12 of them!)

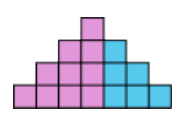
These alternative word problems ask children for much deeper analysis than typical ones, and you can invent them yourself, just by modifying word problems you already have.

> Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. —CCSS

Certain mathematically powerful representations—in particular, the number line, arrays and the area model for multiplication/division, and tabular/spreadsheet forms—are valuable complements to the use of concrete objects in elementary school, and remain important and faithful images of the mathematics through high school.

2. Reason abstractly and quantitatively.

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. —CCSS



This wording sounds very high-schoolish, but the same mathematical practice can be developed in elementary school. Second graders who are learning how to write numerical expressions may be given the challenge of writing numerical expressions that describe the number of tiles in this figure in different ways. Given experience with similar problems so that they know what is being asked of them, students might write 1+2+3+4+3+2+1 (the heights of the stairsteps from left to right) or 1+3+5+7 (the width of the layers from top to bottom) or 10+6 (the number of each color) or various other expressions that capture what they see. These are all decontextualizations—representations that preserve some of

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the original structure of the display, but just in number and not in shape or other features of the picture. Not any expression that totals 16 makes sense—for example, it would seem hard to justify 2+14—but a child who writes, for example, 8+8 and *explains* it as “a sandwich”—the number of blocks in the middle two layers plus the number of blocks in the top and bottom—has taken an abstract idea and added contextual meaning to it.

More generally, Mathematical Practice #2 asks students to be able to translate a problem situation into a number sentence (with or without blanks) and, after they solve the *arithmetic* part (any way), to be able to recognize the connection between all the elements of the sentence and the original problem. It involves making sure that the units (objects!) in problems make sense. So, for example, in *decontextualizing* a problem that asks how many busses are needed for 99 children if each bus seats 44, a child might write 99÷44. But after calculating 2r11 or 2¼ or 2.25, the student must *recontextualize*: the context requires a whole number answer, and not, in this case, just the nearest whole number. Successful recontextualization also means that the student knows that the answer is 3 *busses*, not 3 children or just 3.

3. Construct viable arguments and critique the reasoning of others.

> Mathematically proficient students … justify their conclusions, communicate them to others, and respond to the arguments of others. They … distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is.
>
> … Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. … Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. —CCSS

As every teacher knows, children love to talk. But *explanation*—clear articulation of a sequence of steps or even the chronology of events in a story—is very difficult for children, often even into middle school. To “construct a viable argument,” let alone understand another’s argument well enough to formulate and articulate a logical and constructive “critique,” depends heavily on a shared context, especially in the early grades. Given an interesting task, they can *show* their method and “narrate” their demonstration. Rarely does it make sense to have them try to describe, from their desks, an articulate train of thought, and even more rarely can one expect the other students in class to “follow” that lecture any better than—or even as well as—they’d follow the train of thought of a teacher who is just talking without illustrating. The standard recognizes this fact when it says “students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.” The key is not the concreteness, but the ability to situate their words in context—to show as well as tell.

To develop the reasoning that this standard asks children to communicate, the mathematical tasks we give need depth. Problem that can be solved with only one fairly routine step give students no chance to assemble a mental sequence or argument, even non-verbally. The inclination to “justify their conclusions” also depends on the nature of the task: certain tasks naturally pull children to explain; ones that are too simple or routine feel unexplainable. Depending on the context, “I added” can seem to a child hardly worth saying. And finally, *skill* at “communicating [a justification] to others” comes from having plentiful opportunities to do so. The way children learn language, including mathematical and academic language, is by *producing* it as well as by hearing it used. When students are given a suitably challenging task and allowed to work on it together, their natural drive to communicate helps develop the academic language they will need in order to “construct viable arguments and critique the reasoning of others.”

One kind of task that naturally “pulls” children to explain is a “How many ways can you…” task.

> How many ways can you make 28¢? (Variant: How many ways can you make 28¢ without using dimes or quarters?)
>
> How many different 5"-tall towers of 1" cubes can be made, using exactly one white cube and four blue cubes? (Variant: How many different 5"-tall towers of 1" cubes can be made, using exactly *two* white cubes and three blue cubes?)

The first time young students face problems like these, they tend to be unsystematic. But after they have worked problems like this two or three times, they tend to develop methods (not necessarily efficient or correct, though often so). Then, faced with the question "How can you be sure there are no more?" most children, even as young as second grade, are drawn to explain and do so readily.

Similarly, in the playful context of an imaginary island with two families—one that always tells the truth, and one whose statements are always false—students can hardly stop themselves from explaining how they get answers to questions like this:

> You meet Adam and Beth. Adam says "We're both from the family of liars." Which family is Adam from? What about Beth?

Children (and adults) typically find it far easier to solve the puzzle than to say how they solved it, but it's also typical for them, given the slightest "how'd you get that?," to feel compelled to explain!

While young students can sometimes detect illogical arguments, it is not generally sensible to ask young students to critique the reasoning of others, as it is often too hard for them to distinguish flaws in the logic of another student's argument from artifacts created by the difficulty all young students have in articulating their thinking without ambiguity.

4. Model with mathematics.

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life…. In early grades, this might be as simple as writing an addition equation to describe a situation…. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation…. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs…. They…reflect on whether the results make sense…. —CCSS

The intent of this standard is not to pretend that "problems arising in [the] everyday life" of an adult would be of educational value, let alone interest, to a child. The *serious* everyday-life problems that children face must be solved by the adults who take care of them. Children, themselves, pay closest attention to learning how the world around them works. What we call "play" is their work: they experiment, tinker, push buttons (including ours), say and ask whatever comes to mind, all in an attempt to see what happens. And their curiosity naturally includes ideas we call mathematics: thinking about size, shape and fit, quantity, number. Figuring out where to be to catch a ball is not a paper-and-pencil calculation, but it certainly does involve attention to and rough quantification of the speed, direction, and position of both the ball and oneself, as well as other mathematical ideas. Doing a KenKen® puzzle is also an application of raw arithmetic skills to an everyday life problem, in the honest sense that puzzling is very much an important part of the life of a child (and important enough even in the lives of adults to assure that even supermarkets stock puzzle books).

One intent of this standard is to ensure that children see, even at the earliest ages, that mathematics is not just a collection of skills whose only use is to demonstrate that one has them. Even puzzles suffice for that goal.

Another intent is to ensure that the mathematics students engage in helps them see and interpret the world—the physical world, the mathematical world, and the world of their imagination—through a mathematical lens. One way, mentioned in the standard, is through the use of simplifying assumptions and approximations. Children typically find "estimation" pointless, and even confusing, when they can get exact answers, but many mathematical situations do not provide the information needed for an exact calculation. The following problems suggest two ways children might encounter situations like these in elementary school.

> About how many children are in our school? 50? 200? 1000? To figure that out, we could count, but that's a lot of work. Besides, we don't need to know exactly. How can we come reasonably close, just sitting here in our classroom?
>
> Sam Houston Elementary School has nearly 1,000 children from kindergarten through 5th grade, with about the same number of students in each grade. No class has more than 25 students, but most classes are close to that. What can you figure out from this information?[1]

Both of these examples require "assumptions and approximation to simplify" and also the essential step of "reflect[ing] on whether the results make sense."

What's important here is not the context that's used, but the kind of thinking it requires. Using "approximations to simplify a complicated situation" can be valuable even within mathematics and even when exact answers are required. For example, students in 4th grade who are beginning to learn that there are many fractions equivalent to ½ can quickly become competent, and inventive, at contributing entries to a table, on the board, with three columns: fractions between 0 and ½, fractions equal to ½, and fractions between ½ and 1. They can then apply what they know to simplify, cleverly, such "naked arithmetic" problems such as "Arrange the fractions $^4/_9$, $^5/_8$, and $^7/_{12}$ in order from least to greatest." The comparison can be performed entirely without calculating by noting, first, that $^4/_9$ is less than ½ (because 4 is less than half of 9) and it is the only one that is less than ½, so it is the smallest. The other two are both greater than ½, but $^5/_8$ is one eighth greater than ½ whereas $^7/_{12}$ is only one twelfth greater, so $^5/_8$ is the largest. The point, of course, is not to replace one technique with another. The point is that mathematical thinking simplifies the work.

A completely different kind of modeling involves spatial location: a map or diagram models the real thing. Even children in K and 1 can lay out strips of paper in a grid on the floor, name the streets and avenues, place houses and schools and libraries at various locations, and describe the distances and directions to get from one to another.

5. Use appropriate tools strategically.

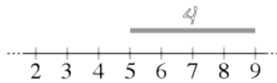
> Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet…. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. —CCSS

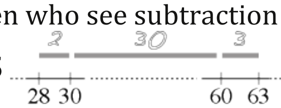
Counters, base-10 blocks, Cuisenaire® Rods, Pattern Blocks, measuring tapes or spoons or cups, and other physical devices are all, if used strategically, of great potential value in the elementary school classroom. They are the "obvious" tools. But this standard also includes "pencil and paper" as a tool, and Mathematical Practice Standard #4 (above) augments "pencil and paper" to distinguish within it "such tools as diagrams, two-way tables, graphs." The number line and area model of multiplication are two more tools—both diagrammatic representations of mathematical structure—that the CCSS Content Standards explicitly require. So, in the context of elementary mathematics, "use appropriate tools strategically" must be interpreted broadly and sensibly to include many choice options for students.

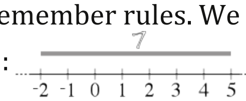
Essential, and easily overlooked, is the call for students to develop the ability "to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained

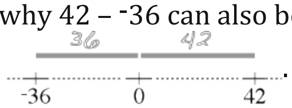
[1] Adapted from *Think Math!*, grade 5.

and their limitations." This certainly requires that students gain sufficient competence with the tools to recognize the differential power they offer; it also requires that their learning include opportunities to *decide for themselves* which tool serves them best. It also requires curricula and teaching to include the kinds of problems that genuinely favor different tools. It may also require that, from time to time, a particular tool is prescribed—or proscribed—until students develop a competency that would allow them to make "sound decisions" about which tool to use.

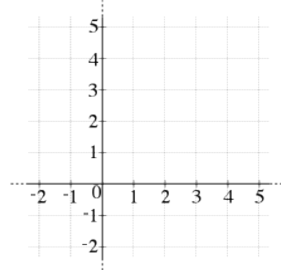
The number line is sometimes regarded just as a visual aid for children. It is, in fact, a sophisticated image used even by mathematicians. For young children, it helps develop early mental images of addition and subtraction that connect arithmetic with measurement. Rulers are just number lines built to spec! This number line image 

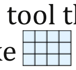
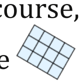
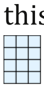
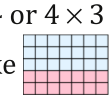
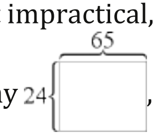
shows "the distance from 5 to 9" or "how much greater 9 is than 5." Children who see subtraction that way can use this model to see "the distance between 28 and 63" as 35 

, and to do so without crossing out digits and borrowing and following a rule they may only barely understand. In fact, many can learn to see this model in their heads, too, and do this subtraction mentally. This is essentially how clerks used to "count up" to make change. The number line model also extends naturally to decimals and fractions by "zooming in" to get a more detailed view of that line between the whole numbers. And it extends equally naturally to negative numbers. It thereby unifies arithmetic, making sense of what is otherwise often seen as a collection of independent and hard-to-remember rules. We can *see* that the distance from $^{-}2$ to 5 is the number we must add to $^{-}2$ to get 5: 

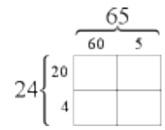
. And we can *see* why 42 – $^{-}36$ can also be written as 42 + 36: the "distance from $^{-}36$ to 42 is 

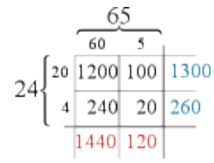
. The number line remains useful as students study data, graphing, and algebra: two number lines, at right angles to each other, label the addresses of points on the coordinate plane.

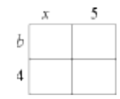


The area model of multiplication is another powerful tool that lasts from early grades through college mathematics. Images like  along with appropriate questions like "how many columns, how many rows, how many little squares" help establish the small multiplication facts. So might pure drill, of course, but this array image goes much further. Seeing the same array held in different positions like  and  makes clear that we can label any of these 3 × 4 or 4 × 3 and the number of little squares is always 12. In grades 3 through 5, array pictures like  help clarify the distributive property of multiplication. This is the property that makes multi-digit multiplication possible, makes sense of the standard multiplication and division algorithms, and underlies the multiplication that students will encounter in algebra. In this picture, we *see* that "two 7s plus three 7s is five 7s." The conventional notation of the idea, $2 \times 7 + 3 \times 7 = (2 + 3) \times 7$, can be a useful, even informative, summary *after* students already understand: a conclusion, rather than a starting place. A schematic version of this image—the area model of multiplication—organizes students' thinking as they learn multi-digit multiplication. In the 3×4 array, counting the squares was not impractical, though *remembering* the fact was certainly more convenient. But in a 65×24 array 

, neither a







memorized fact nor counting are practical. Instead, by partitioning the array, we get a set of steps for which a combination of memorized facts and an understanding of place value help. This image, combined with a spreadsheet-like summary, models the conventional algorithm exactly, making total sense of what can otherwise feel like an arbitrary set of steps. The same image also allows students to acquire and understand the algorithm for division as a process of "undoing" multiplication, greatly simplifying the learning of a part of arithmetic that has a long history of being difficult. What makes this a *powerful* tool is that it serves the immediate goals of elementary school arithmetic in a way that prepares students for algebra. Algebraic multiplication of $(x + 5)(b + 4)$ which has no "carry" step, is modeled perfectly by exactly the same tool. And for 147×46 or for $(y + s - 5)(q - 2)$, no new image is needed. We just extend the "area" model to include the extra terms.

These versatile tools build *mental* models that last. What makes a tool like the number line or area model truly powerful is that it is not just a special-purpose trick or temporary crutch, but is faithful to the mathematics and is extensible and applicable to many domains. These tools help students make sense of the mathematics; that's *why* they last. And that is also why the CCSS mandates them.

6. Attend to precision.

> Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently…. In the elementary grades, students give carefully formulated explanations to each other…. —CCSS

The title is potentially misleading. While this standard does include "calculate accurately and efficiently," its primary focus is *precision of communication*, in speech, in written symbols, and in specifying the nature and units of quantities in numerical answers and in graphs and diagrams.

The mention of definitions can also be misleading. Elementary school children (and, to a lesser extent, even adults) almost never learn new words effectively *from* definitions. Virtually *all* of their vocabulary is acquired from use in context. Children build their own "working definitions" based on their initial experiences. Over time, as they hear and use these words in other contexts, they refine their working definitions and make them more precise. For example, the toddler's first use of "doggie" may refer to all furry things, and only later be applied to a narrower category. In mathematics, too, children can work with ideas without having started with a precise definition. With experience, the concepts will become more precise, and the vocabulary with which we name the concepts will, accordingly, carry more precise meanings. Formal definitions generally come last.

Children's use of language varies with development, but typically does not adhere to "clear definition" as much as to holistic images. That is one reason why children who can *state* that a triangle is a closed figure made up of three straight sides may still choose  as a better example of a triangle than  because it conforms more closely to their mental *image* of triangles, despite its failure to meet the definition they gave.

Curriculum and teaching must be meticulous in the use of mathematical vocabulary and symbols. For example, when students first see the = sign, it may be used in equations like 5 = 3 + 2, or in contexts like 9 + ____ = 8 + 2, in each case making clear that it signals the *equality of expressions*, and is not merely heralding the arrival of an answer. Teacher Guide information about vocabulary must be clear and correct, and must help teachers understand the *role* of vocabulary in clear communication: sometimes fancy words distinguish meanings that common vocabulary does not, and in those cases, they aid precision; but there are also times when fancy words camouflage the meaning. Therefore, while teachers and curriculum should never be sloppy in communication, we should choose our level of precision strategically. The goal of *precision* in communication is *clarity* of communication.

Communication is hard; precise and clear communication takes years to develop and often eludes even highly educated adults. With elementary school children, it is generally less reasonable to expect them to "*state* the meaning of the symbols they choose" in any formal way than to expect them to *demonstrate* their understanding of appropriate terms through unambiguous and correct use. If the teacher and curriculum serve as the "native speakers" of Clear Mathematics, young students, who are the best language learners around, can learn the language from them.

7. Look for and make use of structure.

> Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property…. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects….
> —CCSS

Children naturally seek and make use of structure. It is one of the reasons why young children may say "foots" or "policemans," which they have never heard from adults, instead of *feet* or *policemen*, which they do hear. They induce a structure for plurals from the vast quantity of words they learn and make use of that structure even where it does not apply.

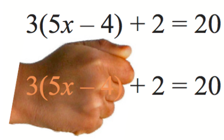
Mathematics has far more consistent structure than our language, but too often it is taught in ways that don't make that structure easily apparent. If, for example, students' first encounter with the addition of same-denominator fractions drew on their well-established spoken structure for adding the counts of things—two sheep plus three sheep makes five sheep, two hundred plus three hundred makes five hundred, and two wugs plus three wugs makes five wugs, no matter what a wug might be—then they would already be sure that two eighths plus three eighths makes five eighths. Instead, they often first encounter the addition of fractions in writing, as $^2/_8 + {}^3/_8$, and they therefore invoke a different pattern they've learned—add everything in sight—resulting in the incorrect and nonsensical $^5/_{16}$. Kindergarteners who have no real idea how big "hundred" or "thousand" are (though they've heard the words) are completely comfortable, amused, and proud to add such big numbers as "two thousand plus two thousand" when the numbers are *spoken*, even though children a year older might have had no idea how to do "2000 + 3000" presented on paper.

This CCSS standard refers to students recognizing that "7 × 8 equals the well-remembered 7 × 5 + 7 × 3." Array pictures help (see MP standard 5, "Use appropriate tools strategically"), but so does students' linguistic knowledge, if the connection is made. The *written* symbols 5×7 + 3×7 = 8×7 are very compact, but the meaning they condense into just eleven characters is something that students understood well even before they learned multiplication. Before they have any idea what a collection of sevens is, they know that five of them plus three of them equals eight of them. It's just five wugs plus three wugs again.

"Standard arithmetic" can be taught with or without attention to pattern. The CCSS acknowledges that students *do* need to know arithmetic facts, but random-order fact drills rely on memory alone, where patterned practice can develop a sense for structure as well. Learning to add 8 to anything—not just to single digit numbers—by thinking of it as adding 10 and subtracting 2 can develop just as fast recall of the facts as random-order practice, but it also allows students quickly to generalize and add 18 or 28 to anything mentally. The *structure* is a general one, not just a set of memorized facts, so students can use it to add 19 or 39, or 21 or 41, to anything, too. With a bit of adjustment, they can use the same thinking to subtract mentally. This is, of course, exactly the way we hope students will mentally perform 350 – 99.

Structure allows sensible definition of odd and even: pairs with or without something left over.

In elementary school, attention to structure also includes the ability to defer evaluation for certain kinds of tasks. For example, when presented with 7 + 5 ◯ 7 + 4 and asked to fill in <, =, or > to compare the two expressions, second graders are often drawn—and may even be explicitly told—to perform the calculations first. But this is a situation in which we want the students' attention on the *structure*, ♣ + 5 ◯ ♣ + 4 or even  + 5 ◯  + 4, rather than on the arithmetic. Without any reference to symbols "standing for" numbers, which might well be distracting or even confusing to second graders, they readily see that  + 5 ◯  + 4 if the same number is under each hand. This same skill of deferring evaluation—putting off calculation until one sees the overall structure—helps students notice that they don't have to find common denominators for 1¾ – ⅓ + 3 + ¼ – ⅔ but can simply rearrange the terms to make such a trivial computation that they can do it in their heads. When students begin to solve algebraic equations, the same idea will help them notice that $3(5x - 4) + 2 = 20$ can be treated as "something plus 2 equals 20" and conclude, using common sense and not just "rules," that $3(5x - 4) = 18$.



From such reasoning, they can learn to *derive* rules that make sense.

8. Look for and express regularity in repeated reasoning.

> Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal…. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results. —CCSS

A central idea here is that mathematics is open to drawing general results (or at least good conjectures) from trying examples, looking for regularity, and describing the pattern both in what you have done and in the results.

A multiplication fact-practice exercise might, for example, ask children to choose three numbers in a row (e.g., 5, 6, and 7) and compare the middle number times itself to the product of the two outer numbers. In this example, the two products are 36 and 35. After they do this for several triplets of numbers, they are likely to conjecture a pattern that allows them to multiply 29 × 31 mentally because they expect it to be one less than 30 × 30, which they can do easily in their heads. Seeing that regularity is typically easy for fourth graders; expressing it clearly is much harder. Initial attempts are generally inarticulate until students are given the idea of *naming* the numbers. A simple non-algebraic "naming" scheme was used above to describe the pattern: the numbers were named "middle" and "outer" and that was sufficient. A slightly more sophisticated scheme would distinguish the outer numbers as something like "middle plus 1" and "middle minus 1." Then children can state (middle – 1) × (middle + 1) = middle^2 – 1. The step from this statement to standard algebra is just a matter of adopting algebraic conventions: naming numbers with a single letter like *m* instead of a whole word like "middle," and omitting the × sign.

The recognition that adding 9 can be simplified by treating it as adding 10 and subtracting 1 can be a discovery rather than a taught strategy. In one activity—there are obviously many other ways of doing this—children start, e.g., with 28 and respond as the teacher repeat only the words "ten more" (38), "ten more" (48), "ten more" (58), and so on. They may even be counting, initially, to verify that they are actually adding 10, but they soon hear the pattern in their responses (because no other explanatory or instruction words are interfering) and express that discovery from their repeated reasoning by saying the 68, 78, 88 almost without even the request for "ten more." When, at some point, the teacher changes and asks for "9 more," even young students often see it as "almost ten more" and make the correction spontaneously. Describing the discovery then becomes a case of "expressing regularity" that was found through "repeated reasoning." Young students then find it very exciting to add 99 the same way, first by repeating the experience of getting used to a simple computation, adding 100, and then by coming up with their own adjustment to add 99.

Differences between, and connections between, Content and Practice standards

> **Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content**
>
> The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction…. —CCSS

Standards and curricula are different objects. In order to achieve a standard solidly and with full fidelity, curricula and teaching must sometimes take preparatory steps that are not specified in the standards. Such divergence cannot omit either the content or the practice standards and remain faithful; neither can it crowd classroom instruction with material that the CCSS deliberately pruned out of the "mile wide, inch deep" curriculum in an attempt to achieve depth and focus. But fidelity to both Content and Practice requires more than a checklist approach: a coherent mathematical story line cannot be achieved by teaching to the standards any more than by teaching to the test. In part, that is the message of the concluding statement in the Mathematical Practice section that calls for connecting those standards to the standards for Mathematical Content.

Standards can mandate this connection, as the CCSS does, but specifying its design is a job for curriculum materials. Mathematical facts and procedures—the Content part of what we teach—are the results of the application of mathematical habits of mind reflected in the Practices. For that reason, fidelity to the way mathematics is made and used—a big part of the intent of the Mathematical Practices—requires that the Content be taught *through* the Practices. That way, the connections are *real*—integrated rather than interspersed.

This attention to Mathematical Practices *connected* with Content must also be enacted in teaching, which will require professional development. Though the CCSS Mathematical Content standards differ in detail from other content standards, their form is familiar to teachers: a list of things to know. The Mathematical Practices are not so easily condensed into a lesson or unit, not so easily tested and, generally, not so familiar. Content standards are specified grade by grade and build on each other rather than repeating year after year. The Mathematical Practices are different. Though they can be enacted in an appropriate way at any level, they evolve and mature over years rather than days, along with children's cognitive development and the nature and sophistication of the Mathematical Content.

Finally, of course, any system that rewards and/or punishes teachers and schools on the basis of measured results can expect that anything that isn't reflected in the measuring instruments will be ignored completely. The Mathematical Practices will be taken seriously in curriculum and teaching if, and only if, they are taken seriously in testing. It can be expected, then, that the developers of the CCSS, and the States that collaborated in calling for the development of the CCSS, will work with the developers of assessments to ensure that the Mathematical Practices are taken seriously in testing.

