Early Algebra and the Common Core: What Do Teachers Need to Know?

Susan Jo Russell
Deborah Schifter

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Standards for Mathematical Content

- Grade 1: Apply properties of operations as strategies to add and subtract.

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- Grade 1: Apply properties of operations as strategies to add and subtract.
- Grade 2: Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
- Grade 3: Identify arithmetic patterns and explain them using the properties of the operations.

Standards for Mathematical Content

- Grade 4: Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations.

- Grade 5: Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division.

In this talk:

- Part 1: Focus on early algebra in the elementary grades—a constellation of content and practice standards
- Part 2: What third graders can do—an example from an experienced teacher's classroom
- Part 3: What do elementary teachers need to know to support students' work on the properties of the operations? Video clips and discussion
- Part 4: Concluding remarks

Part 1: Focus on early algebra in the elementary grades—a constellation of content and practice standards

Foundations of Algebra in the Elementary and Middle Grades

Using Routines as an Instructional Tool for Developing Elementary Students' Conceptions of Proof

Susan Jo Russell
Deborah Schifter
Virginia Bastable

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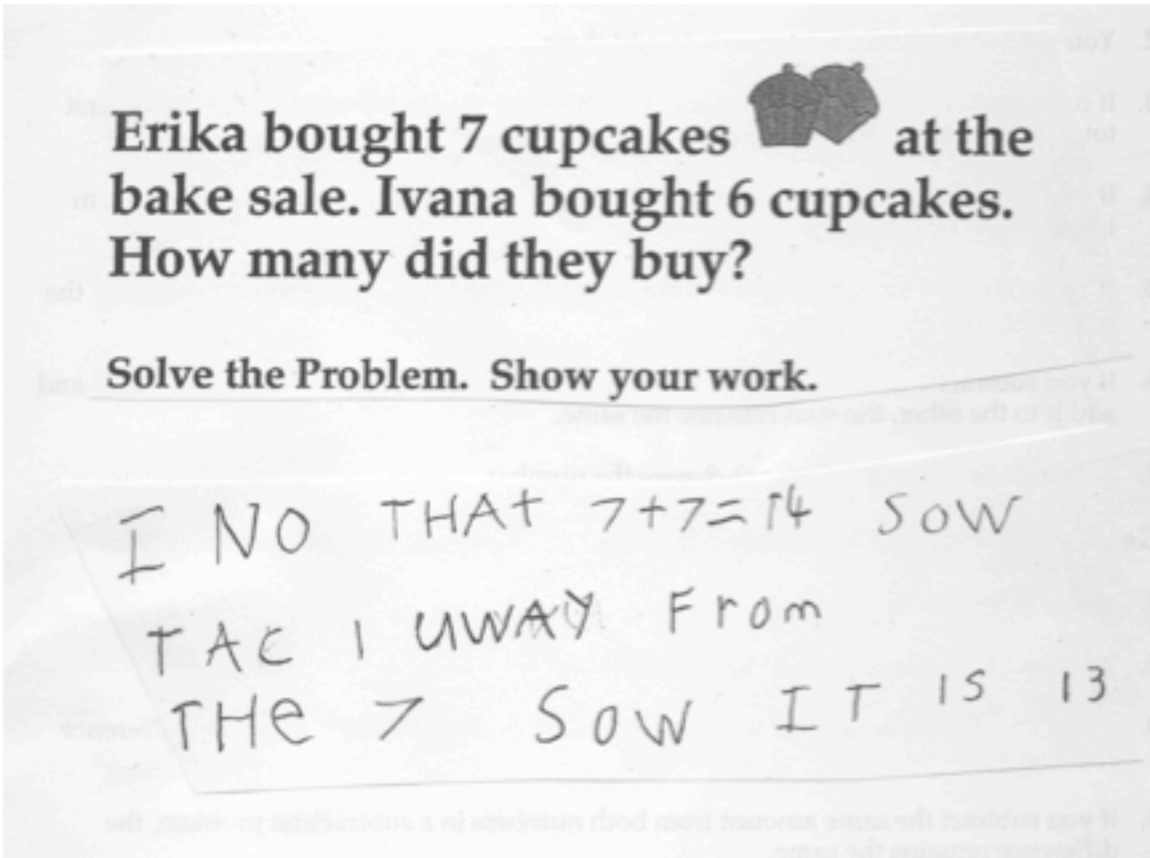
Leah's Work



Erika bought 7 cupcakes at the bake sale. Ivana bought 6 cupcakes. How many did they buy?

Solve the Problem. Show your work.

I NO THAT 7+7=14 SOW
TAC 1 UWAY From
THE 7 SOW IT IS 13

Leah's thinking:

$7 + 7 = 14 \rightarrow 7 + 6 = 13$

A general idea based on Leah's thinking:
In an addition problem,
if an addend is decreased by 1,
then the sum is decreased by 1.

For any numbers, a, b, and c, if $a + b = c$,
then $a + (b - 1) = c - 1$
or $a + (b - 1) = (a + b) - 1$.

Key aspects of integrating early algebra into arithmetic instruction

- Investigating, describing, and justifying general claims about how an operation behaves
- A shift in focus from solving individual problems to looking for regularities and patterns across problems
- Representations of the operations as the basis for proof
- The operations as objects of study in themselves

$$9 + 4 = 10 + 3$$

What is happening when you change the expression 9 + 4 to 10 + 3?

Does this work with other numbers?

Does this work with all numbers? How do you know?

$$9 + 4 = 10 + 3$$

What is happening when you change the expression $9 + 4$ to $10 + 3$?

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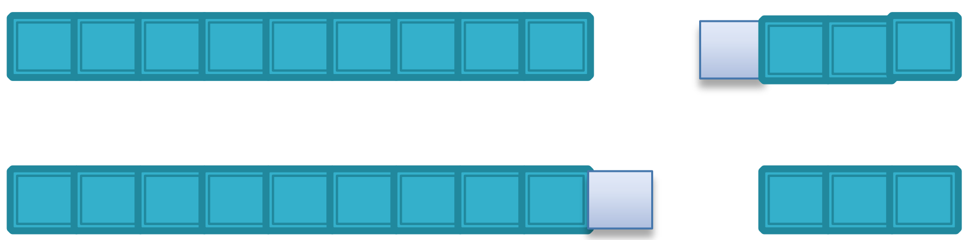
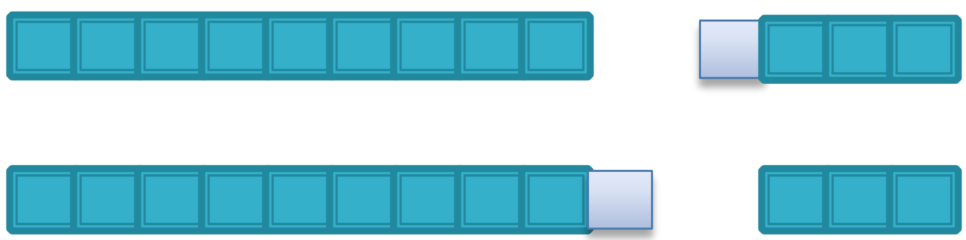


$$9 + 4 = 10 + 3$$

What is happening when you change the expression 9 + 4 to 10 + 3?



$$a + b = (a + n) + (b - n)$$

What happens with subtraction?

Is it true that 9 - 4 = 10 - 3? Why doesn’t it work?

Is there another rule that works for subtraction?

What happens with multiplication?

Is it true that 9 x 4 = 10 x 3?

Why doesn't it work?   What does work?

A constellation of related content and practice standards

- Understand and apply properties of operations
- MP8: Look for and express regularity in repeated reasoning
- MP6: Attend to precision
- MP3: Construct viable arguments and critique the reasoning of others

Middle School Students' Errors

$(3 + 5)^2 \neq 3^2 + 5^2$

but we see middle grade students write:

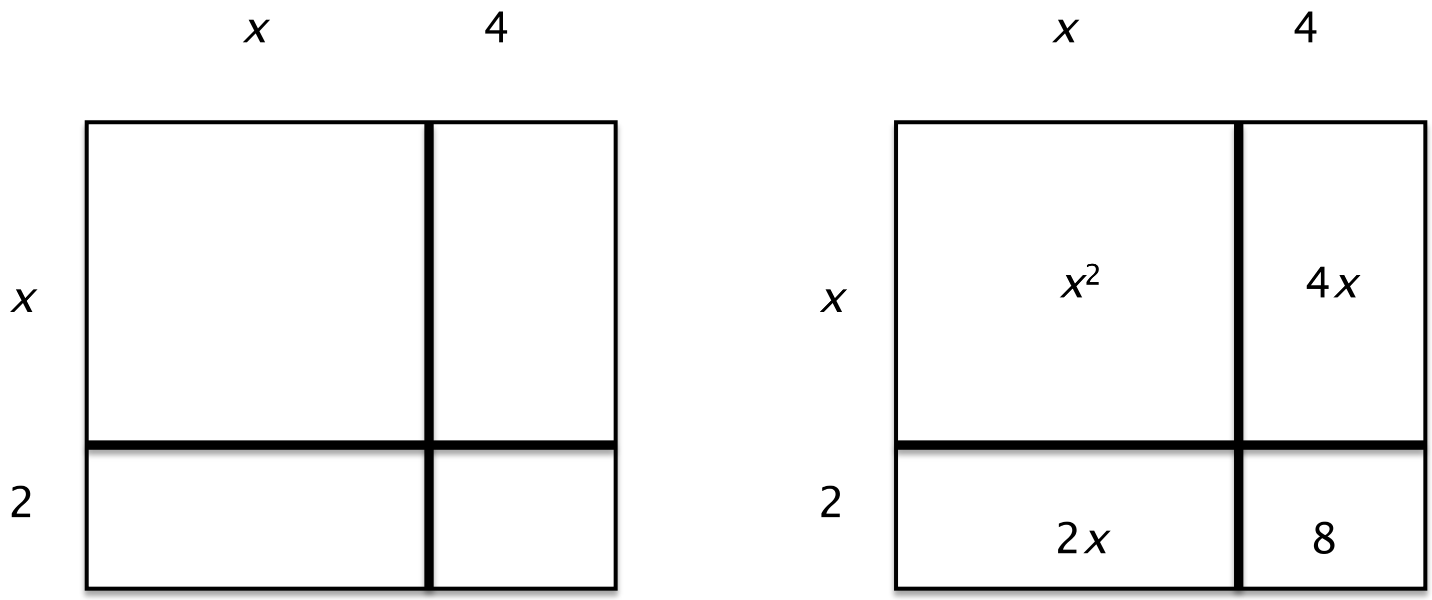
$$(a + b)^2 = a^2 + b^2$$

$2 \times (3 \times 5) \neq (2 \times 3) \times (2 \times 5)$

but we see middle grade students write:

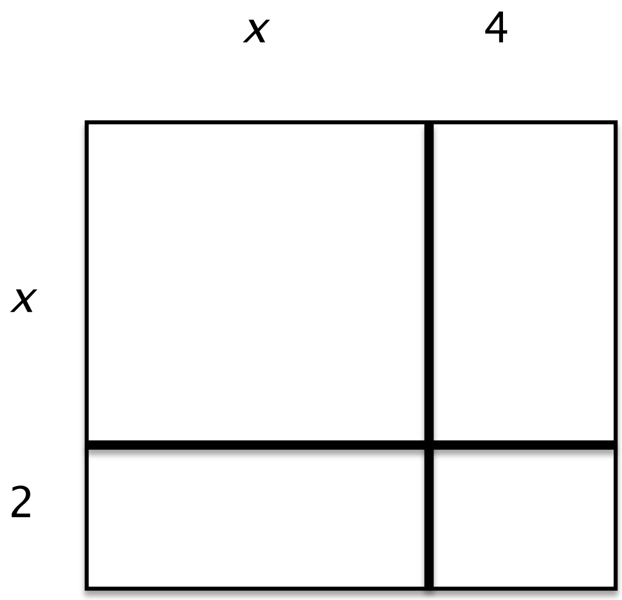
$$2\,(ab) = (2a)(2b)$$

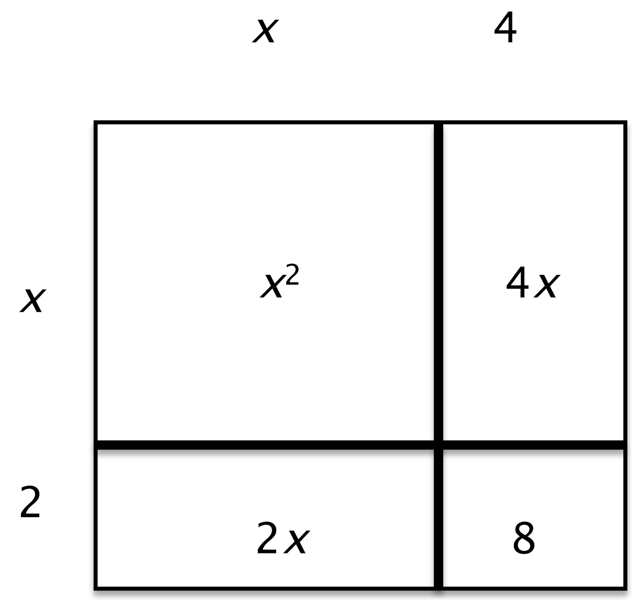
Joy: an 8th grader



$x^2 + 4x + 2x + 8 =$

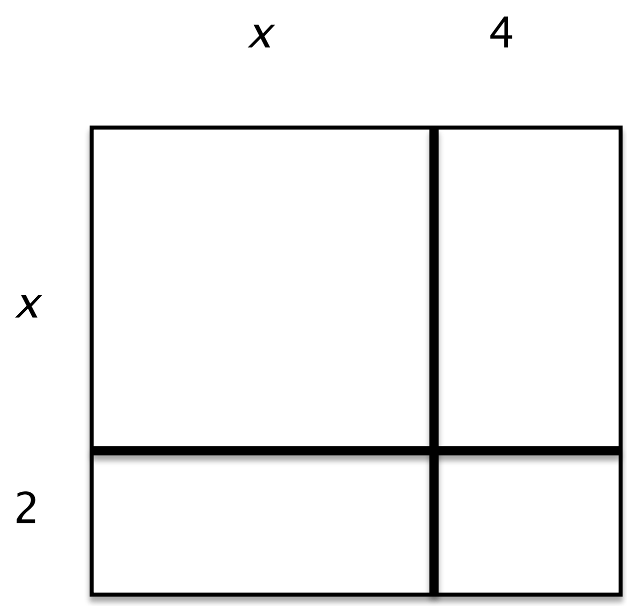
Joy: an 8th grader

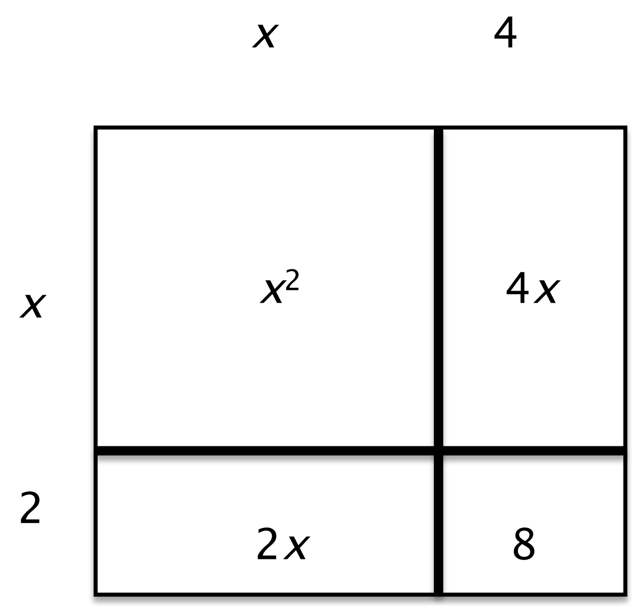




$$x^2 + 4x + 2x + 8 = (x \cdot x) + (2 \cdot 4)$$

Joy: an 8th grader





$$x^2 + 4x + 2x + 8 = (4x)\,(2x)$$

Mr. Lancaster:

I've been wondering for a while what the fundamental difficulties are that students encounter in mathematics at the middle school level. I used to think that the fundamental difficulty had to do with the introduction of variables, and with expressions and formulas.

Mr. Lancaster (con.):

More recently, I've come to think that an even more fundamental difficulty has to do with the understanding of multiplication as distinct from addition. I'm not talking about whether students know their multiplication facts, and I'm not talking about whether students can memorize different sets of rules for things additive as for things multiplicative.

Mr. Lancaster (con.):

The conclusion I'm coming to is that the students who have the deepest trouble with middle school mathematics are those without a clear and rich set of models for what multiplication is and how it is different from addition. In the absence of such models, the students tend to rely increasingly on memorizing rules and how-to's, and their grasp is tentative and fragile. They have given up trying to make sense of what they are doing, and instead rely on their quite well-honed skills of guessing what the teacher wants.

Mr. Lancaster (con.):

To my surprise, this includes many of the students who have decent grades, and who can get by quite well with symbolic manipulation of formulas that may hold very little meaning for them.

Part 2: What third graders can do—an example from an experienced teacher's classroom

A Look at a Third Grade Class

What do you notice about how this teacher focuses her students on:

- Articulating a general claim?
- Representing a general claim?
- Proving a general claim?

How is multiplication different from addition?

What happens to the sum when 1 is added to an addend?

$$3 + 5 \rightarrow 4 + 5$$
$$3 + 5 \rightarrow 3 + 6$$

What happens to the product when 1 is added to a factor?

$$3 \times 5 \rightarrow 4 \times 5$$
$$3 \times 5 \rightarrow 3 \times 6$$

7 + 5 = 12 7 + 6 = __	7 + 5 = 12 8 + 5 = __
9 + 4 = 13 9 + 5 = __	9 + 4 = 13 10 + 4 = __

Articulation of the claim

- In addition, if you increase one of the addends by 1, then the sum will also increase by 1.

- In addition, if you increase one of the addends by 1, and keep the other addend(s) the same, then the sum will also increase by 1.

Teacher’s challenge

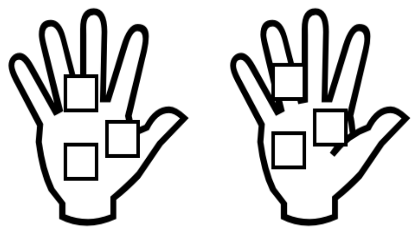
- How would you go about convincing somebody that this statement is true?
- What would you do to convince yourself or someone else?

9,000,000,000,000,000,000,000,000,000,000,000 + 9,000,000,000,000,000,000,000,000,000,000,001 = 18 decillion and 1

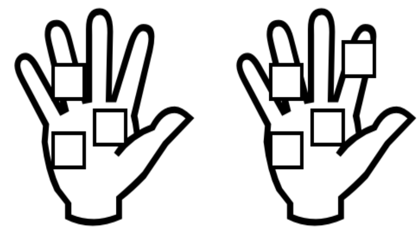
9 decillion and 2 + 9 decillion = 18 decillion and 2

"If it works for numbers of this size, it must always work!"

Megan's poster and explanation



3 + 3 = 6



3 + 4 = 7

The picture could be used for ANY numbers, not just 3 and 4. I could have started with anything in one hand, and then anything else in the other hand, and put them together. If I got 1 more thing in either hand, the total would always only go up by 1.

Adding 1 to a factor

7 x 5 = 35 7 x 6 = 42	7 x 5 = 35 8 x 5 = 40
9 x 4 = 36 9 x 5 = 45	9 x 4 = 36 10 x 4 = 40

Writing prompt. In a multiplication problem, if you add 1 to a factor, I think this will happen to the product:

Choose which of the original equations you want to work with. Then do one of these…

- Draw a picture for the original equation; then change it just enough to match the new equations.
- Make an array for the original equation; then change it just enough to match the new equations.
- Write a story for the original equation; then change it just enough to match the new equations.

Example: Original equation $7 \times 5 = 35$

New equations $7 \times 6 = 42$

$8 \times 5 = 40$

Frannie's story context

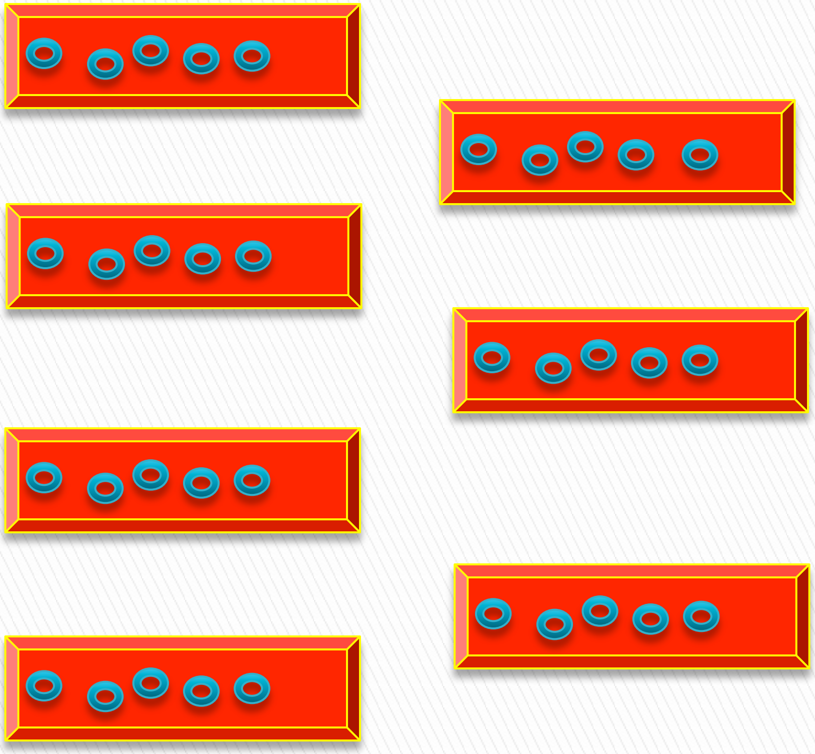
There are 7 jewelry boxes and each box has 5 pieces of jewelry. There are 35 pieces of jewelry altogether.

Jewelry boxes

7 x 5

Seven boxes with five pieces of jewelry in each box

35 pieces of jewelry

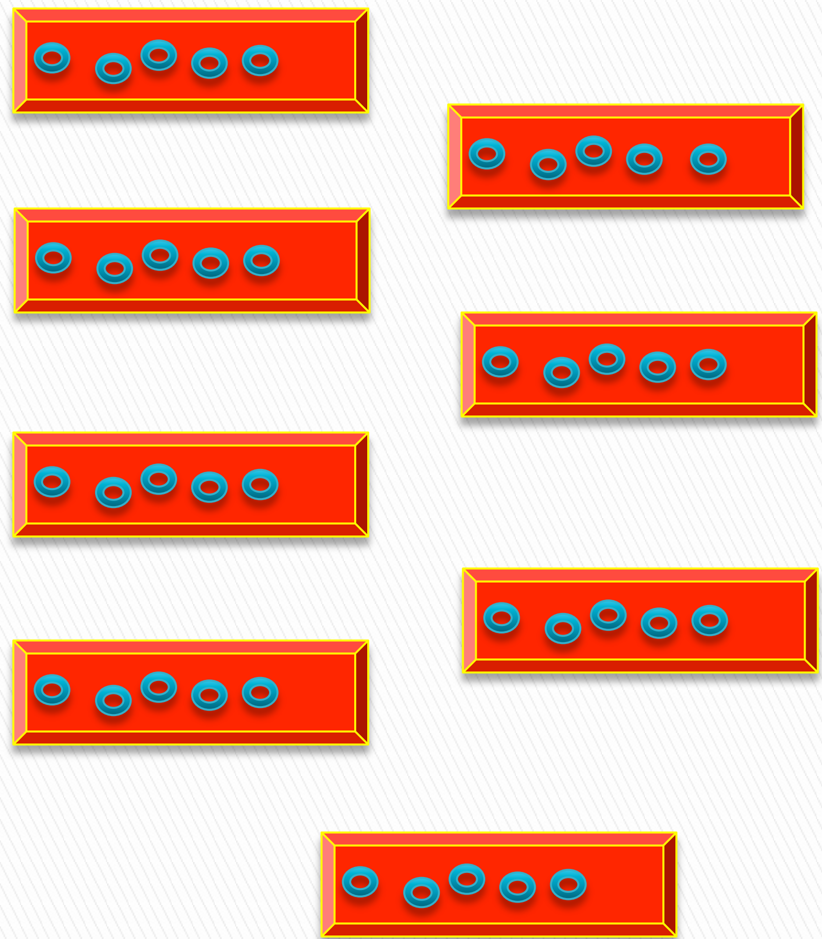


Jewelry boxes

7 x 5 → 8 x 5

Eight
~~Seven~~ boxes with five pieces of jewelry in each box

35 pieces of jewelry
+ 5 pieces of jewelry
40 pieces of jewelry

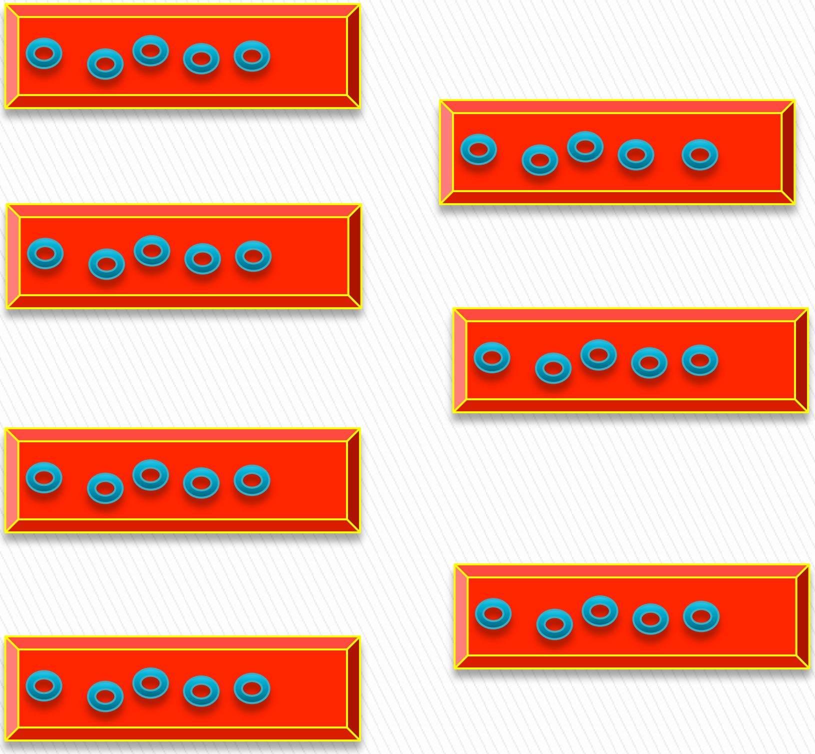


Jewelry boxes

7 x 5

Seven boxes with five pieces of jewelry in each box

35 pieces of jewelry



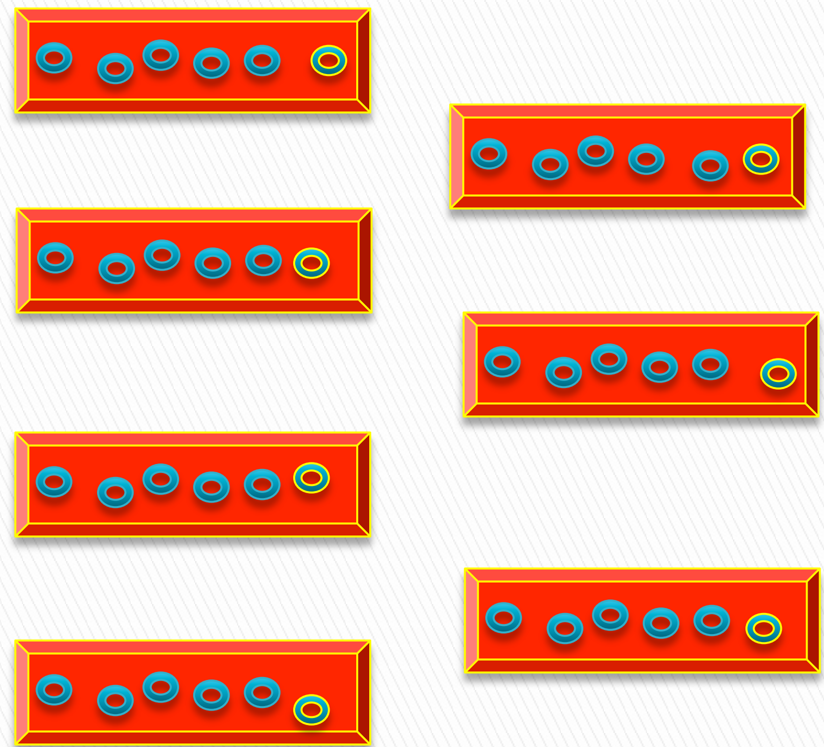
Jewelry boxes

7 x 5 → 7 x 6

six

Seven boxes with ~~five~~ pieces of jewelry in each box

  35 pieces of jewelry
+ 7 pieces of jewelry
  42 pieces of jewelry



Students' articulations of the claim

- The number that is not increased is the number that the answer goes up by.
- The number that is staying and not going up, increases by however many it is.
- I think that the factor you increase, it goes up by the other factor.

Articulating the claim about increasing a factor by 1

$a \times b$

Addison: If you increase the first number by 1, you increase the answer by the second number. Like if you increase *a* by 1, you increase the product by *b*. If you increase *b* by 1, you increase the answer by *a*.

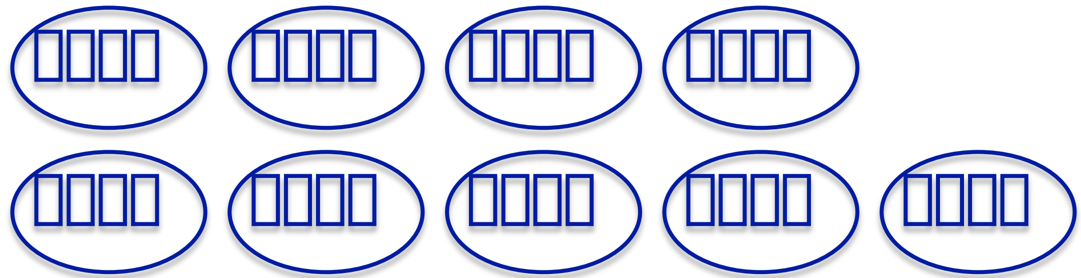
Addison: If you increase the first number by 1, you increase the answer by the second number. Like if you increase a by 1, you increase the product by b. If you increase b by 1, you increase the answer by a.

Ms. Kaye: I love this context! Because you can have any number of jewelry boxes, each one with a certain number of jewels in it. And the product is going to be that multiplication. And if you increase the number of boxes, you get an increase by the number in each box. If you increase by one jewel, the increase is the number of boxes because each box gets one jewel.

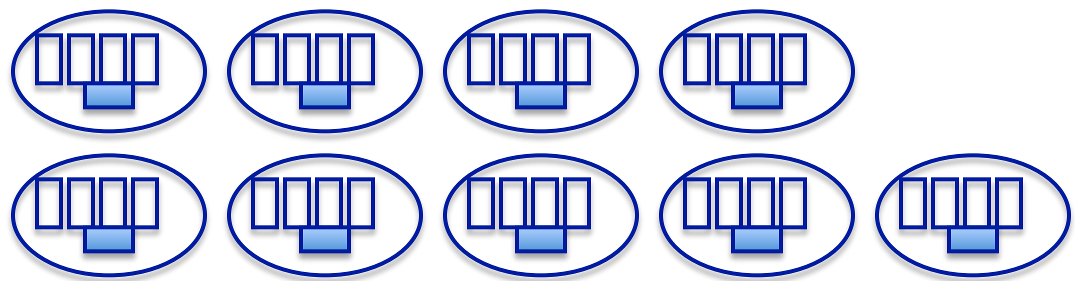
Other stories

- Baskets of bouncy balls
- Tanks with salmon eggs
- Baskets of mozzarella sticks
- Rows of chairs

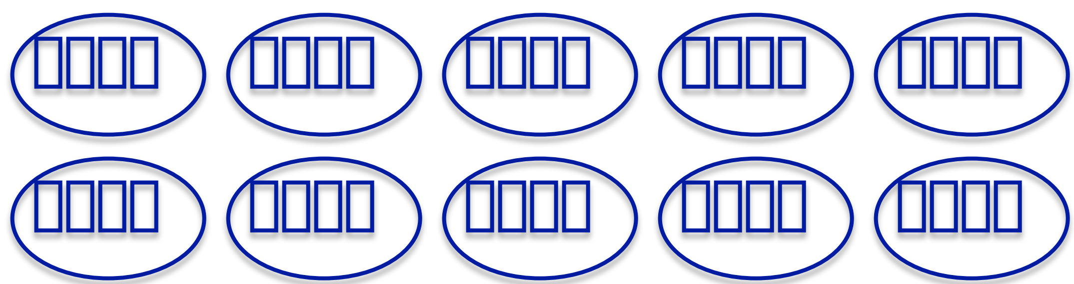
9 x 4 = 36

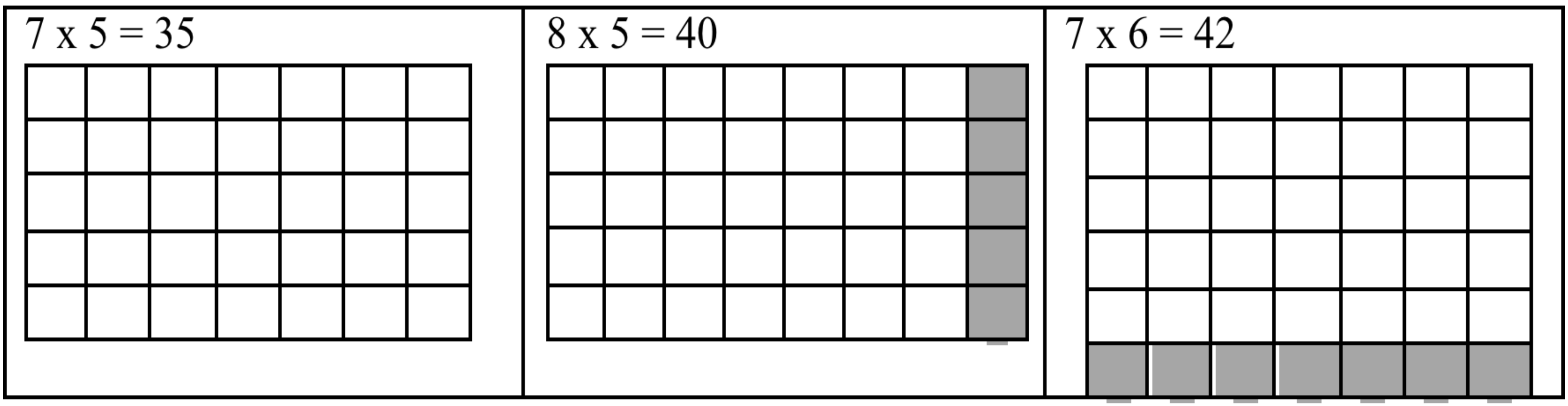


9 x 5 = 45



10 x 4 = 40




7 x 5 = 35
8 x 5 = 40
7 x 6 = 42

Some observations

- Properties of the operations (even special cases of them) are worthwhile objects of inquiry.

- $a + (b + 1) = (a + b) + 1$
- $a(b + 1) = ab + a$
- $(a + 1)b = ab + b$

Some observations

- Properties of the operations (even special cases of them) are worthwhile objects of inquiry.
- There are other generalizations about the behavior of the operations that are worthy of attention.

$$a + b = (a + n) + (b - n)$$

$$a - (b + 1) = (a - b) - 1$$

Some observations

- Properties of the operations (even special cases of them) are worthwhile objects of inquiry.
- There are other generalizations about the behavior of the operations that are worthy of attention.
- Keeping symbols connected to representations is key.

Part 3: What do elementary teachers need to know to support students' work on the properties of the operations?

Video clips and discussion

Components of Investigating a Generalization

- Noticing regularity
- Articulating general claims
- Investigating through representations
- Constructing arguments
- Comparing operations

Investigating the properties of the operations

- What is the work of the elementary teacher?
- What mathematical knowledge is required?

Clip 1: Noticing regularity

First and second graders investigate the relationship between addition and subtraction

- Maeve had 7 red flowers. She also had 19 yellow flowers. How many flowers did she have altogether?
- Peter caught 19 fish in the morning. He caught 7 more in the afternoon. How many fish did he catch all together?
- Elizabeth had 26 balloons. 19 flew away. How many did she have left? Can you use what you know from the previous problems to help you with this problem? How does it help?

Noticing regularity

- What is the work of the elementary teacher in supporting students to notice regularities?

- What mathematical knowledge is required?

Clip 2: Articulating a general claim

Third graders investigate subtraction

Subtracting from the same amount	Subtracting the same amount
10 – 3 10 – 4	10 – 3 9 – 3
25 – 5 25 – 7	25 – 5 23 – 5
30 – 10 30 – 15	30 – 10 20 – 10

Articulating a general claim

- What is the work of the elementary teacher in supporting students to articulate a general claim?

- What mathematical knowledge is required?

Clip 3: Investigating specific cases of a generalization through representations

Fourth graders represent multiplication

How do you know $10 \times 3 = 5 \times 6$?

Investigating specific cases of a generalization through representations

- What is the work of the elementary teacher in supporting students to use representations to investigate specific cases of a generalization?

- What mathematical knowledge is required?

Clip 4: Constructing arguments

Third graders use representations as the basis of an argument

Students have articulated this claim:

"In addition, when one of the addends goes up by 1, the sum always has to go up by 1."

Constructing arguments

- What is the work of the elementary teacher in supporting students to construct arguments?
- What mathematical knowledge is required?

Clip 5: Comparing operations

Second graders compare addition and subtraction

Second graders have been investigating the idea that switching the addends in an addition expression does not change the sum.

Now they are asked: if you switch the numbers of a subtraction expression, does the result still stay the same?

Comparing operations

- What is the work of the elementary teacher in supporting students to compare operations?
- What mathematical knowledge is required?

- Clip 1: First and second graders; relationship between addition and subtraction
- Clip 2: Third graders; articulation of a generalization about subtraction
- Clip 3: Fourth graders; array representation of 10 x 3 and 5 x 6
- Clip 4: Third graders; representation of the generalization about adding 1 to an addend
- Clip 5: Second graders; 3 – 7 is not equal to 7 – 3

Part 4: Concluding remarks

What teachers need to know

- Know which generalizations are fruitful
  - Select and sequence examples
- Be able to hear and interpret what students say and figure out how what they say is connected to the idea being considered
  - Decide when and how to help students articulate their claim
- Be familiar with different representations of the operations—the affordances and limitations of each

What teachers need to know

- Understand what mathematical proof is and what that means for students at their grade level
- Understand the importance of comparing the behavior of different operations
- Keep track of the progress of student ideas

Three pillars of number and operations

- Understanding numbers
- Computational fluency
- Examining the behavior of the operations

Three pillars of number and operations

- Understanding numbers
  - written and oral counting
  - the structure of the base ten system with whole numbers and decimals
  - the meaning of fractions, zero, and negative numbers
- Computational fluency
  - accurate, efficient, and flexible strategies for each operation
  - knowing how and when to apply them

Three pillars of number and operations

- Examining the behavior of the operations
  - modeling these operations
  - learning about the properties of each operation
  - describing and justifying the behaviors that are consequences of those properties
  - comparing and contrasting the behaviors of different operations

Teachers' realizations

- They used to think of the heart of their curriculum as understanding numbers and learning to compute.
- Now they'd identified a third objective of equal weight: investigating the behavior of the operations

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- They used to think they had to develop special lessons to engage students with generalizing about operations.
- Now they see that opportunities to work on this "third pillar" arise in all their work on number and operations.

A constellation of related content and practice standards

- Understand and apply properties of operations
- Look for and express regularity in repeated reasoning
- Attend to precision
- Construct viable arguments and critique the reasoning of others

Is multiplication related to division in the same way as addition is related to subtraction?

You know that thing you can do with addition when you keep one number whole and add the other one on in parts? Well, I was wondering. Can you do that same thing with multiplication?