Teachers' Knowledge for Using Drawn Models of Fraction Arithmetic

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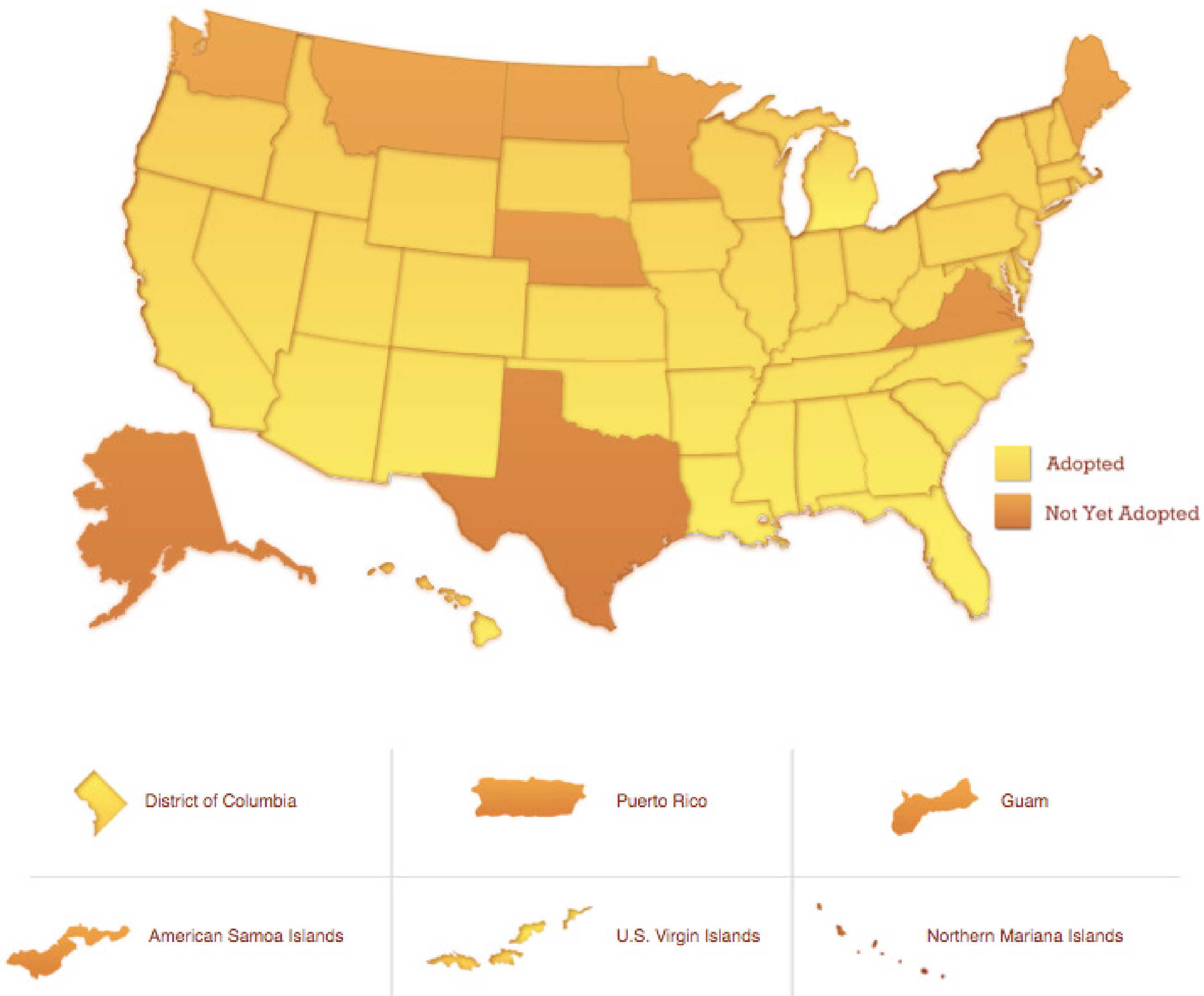
Why Emphasize Drawn Models?

- Drawn models: Inscriptions conveying relationships among numbers or quantities (e.g., number lines, rectangular areas)
- Theories of learning (e.g., Piaget) emphasize experiences in the material world as the basis for students developing mathematical concepts
- Theories of teacher knowledge (e.g., Ball, Shulman) emphasize representations
- Curricular standards (e.g., NCTM, CCSS) assign a central role to representations

Common Core Standards Adoption



Why Emphasize Fractions?

- Essential for the study of algebra (e.g., NMAP, Kilpatrick & Izsák, Wu)
  - Understanding proportional relationships among quantities (e.g., rate of change)
  - Manipulating algebraic notation (e.g., like terms)
  - Working with formal properties of number systems (e.g., deducing general numeric methods)

Why Emphasize Fractions?

- Fraction Division (e.g., Ball, 1990; Borko et al. 1992; Ma, 1999)
  - Give a situation that illustrates $1\frac{3}{4} \div \frac{1}{2}$
  - Generate drawn models for fraction arithmetic
- Decimal Multiplication (e.g., Graeber et al., 1989)
  - 1 kg of detergent makes 15 kg of soap. How much soap does .75 kg of detergent make?

Organization

- Three projects studying teachers' reasoning with drawn models for fraction arithmetic
- Each new project builds on previous project
- Moving from intensive case studies of individual teachers in their classrooms, to groups of teachers in professional development, to national samples
- Harnessing psychometric models as a research tool
- Implications for *Common Core State Standards*

Project 1: Coordinating Students' and Teachers' Algebraic Reasoning

- How do teachers use and build upon their existing knowledge when understanding and responding to mathematical problems that arise during classroom interactions?
- Pierce Middle School
- *Connected Mathematics Project* (CMP)
- Enactment of entire instructional units in Grades 6–8
- Videotaped lessons, student interviews, and teacher interviews

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Case Studies of Two 6th-Grade Teachers: Drawn Models for Fraction Multiplication

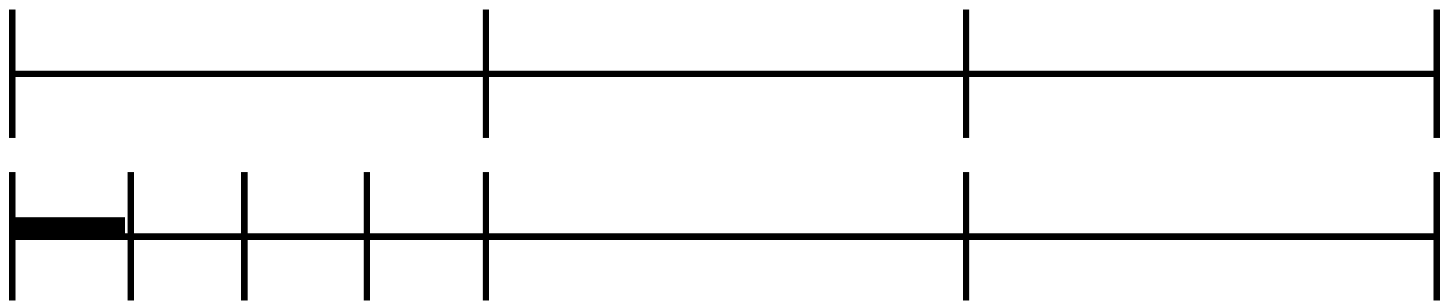
- Izsák (2008)
- Examined moments during instruction when each teacher was more/less flexible when responding to students' thinking
- Generated accounts of each teacher's knowledge
- Explained sequences of lessons spanning several weeks

Knowledge for Teaching Fraction Multiplication–Part 1

Multiplication and Unit Structures

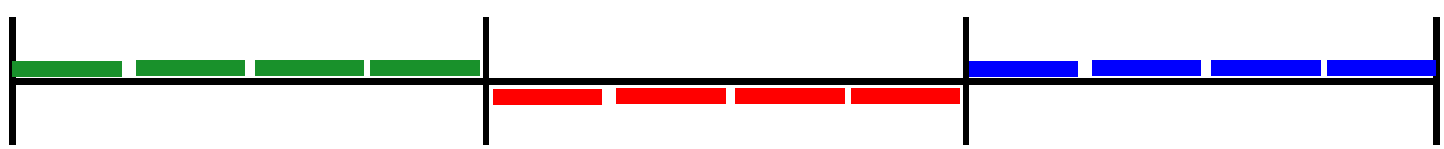
- The algorithm
- Multiplication is repeated addition
- A fraction *of* a number means a fraction *times* the number
- Products of rectangular dimensions give areas
- Unit structures (2- vs. 3-levels of units)
- Drawn instantiations of the distributive property

Levels of Units: What is 1/4 of 1/3?



Solving with 2-levels of units:

Solving with 3-levels of units:



Knowledge for Teaching Fraction Multiplication–Part 2

Pedagogical Uses for Drawings

- *Illustrate* computed solutions
- *Infer* a numeric method from patterns
- *Deduce* a general numeric method from represented structure of quantities
- *Adapt* to students' strategies to generate a general numeric method

Ms. Archer

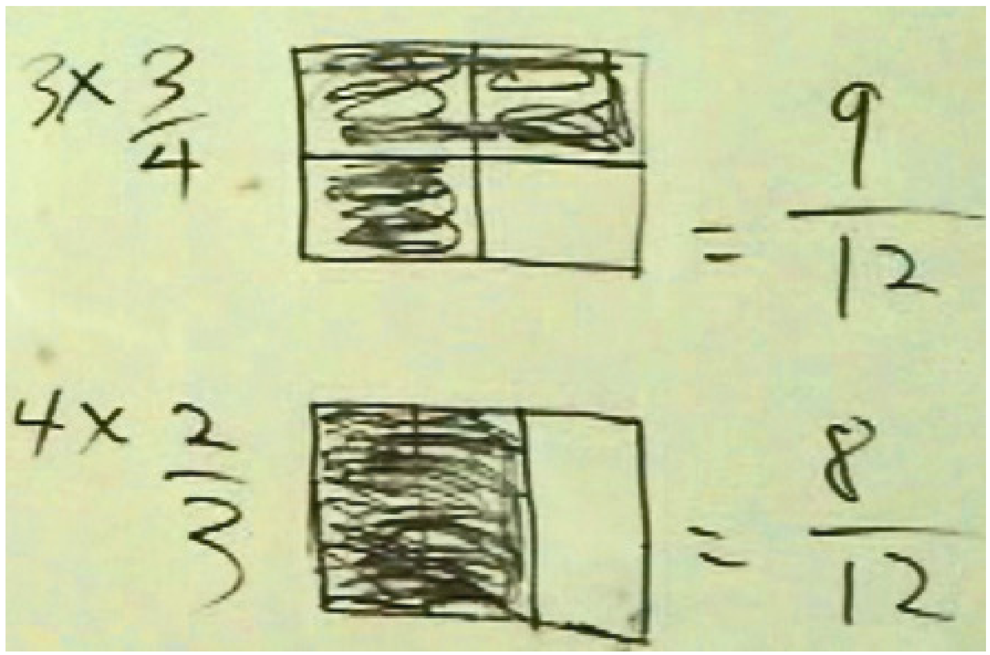
- Used 2-level structures and understandings associated with multiplication to reason about parts of parts and to *illustrate particular solutions*

Ms. Reese

- Used 3-level structures and understandings associated with multiplication to reason about parts of parts and to *infer a general method*

Example of Constraints

- Ms. Archer used areas to compare 3/4 and 2/3
- Reported afterwards she did not think of 12ths



Project 2: Does it Work?

- What do teachers learn from *InterMath* professional development experiences?
- Professional development emphasized fractions and proportions, drawn models, and referent units
- Developed a pretest/posttest aligned to content of professional development

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Fraction Division: Referent Units

- The units to which numbers refer
  - One referent unit for all numbers:

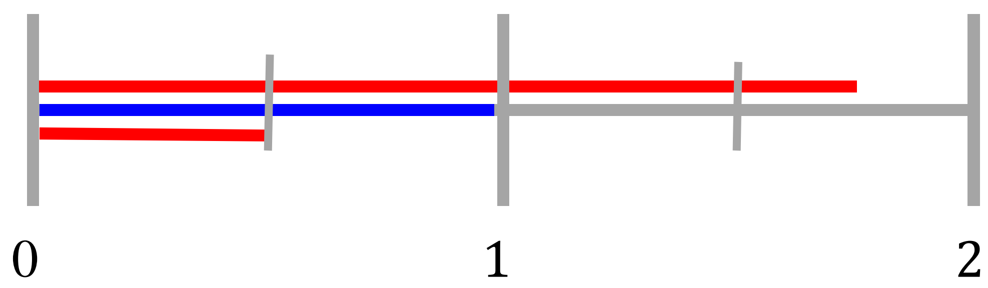
$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$

  - Different referent units for each number:

$$1\frac{3}{4}\div\frac{1}{2}=3\frac{1}{2}$$

Fraction Division: Nested Units

$$1\frac{3}{4} \div \frac{1}{2} = 3\frac{1}{2}$$



The Does it Work? Instrument

- Adapted the Learning Mathematics for Teaching (LMT) middle grades measure of MKT (Hill, 2007)
- Three types of multiple-choice questions
  - **Numeric**: Justify standard numeric procedures, evaluate students' proposed numeric methods
  - **Verbal**: Identify referent units presented verbally (word problems)
  - **Drawing**: Identify referent units presented through drawings

Rational Number Content Matrix

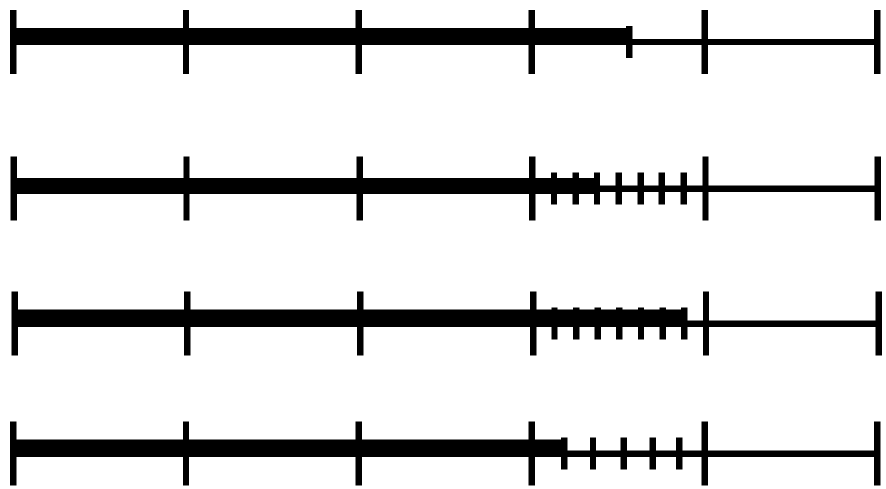
		Numeric	Verbal	Drawing
Fractions	Compare	2	7	1
	Add/Sub	1	4	(2)
	Multiplication	2	2	2 (5)
	Division	1	4	(3)
	Ratio/Proportion	1	1 (4)	(4)
Decimals	Compare	1 (1)	–	–
	Add/Sub	1	–	–
	Multiplication	1	–	4
	Division	1	4	–

Sample Item

Ms. Roland gave her students the following problem to solve:

Candice has 4/5 of a meter of cloth. She uses 1/8 of a meter for a project. How much cloth does she have left after the project?

Which of the following diagrams shows the solution?



1/8 of 4/5

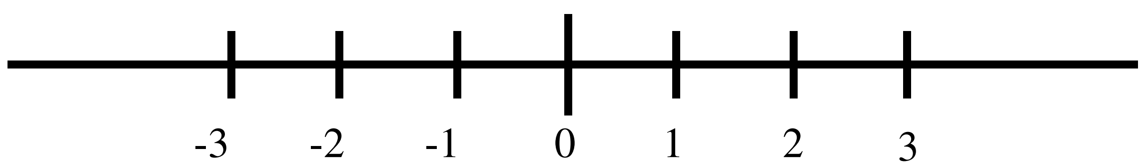
5/40 of 1

1/8 of 1/5

5/30 of 1

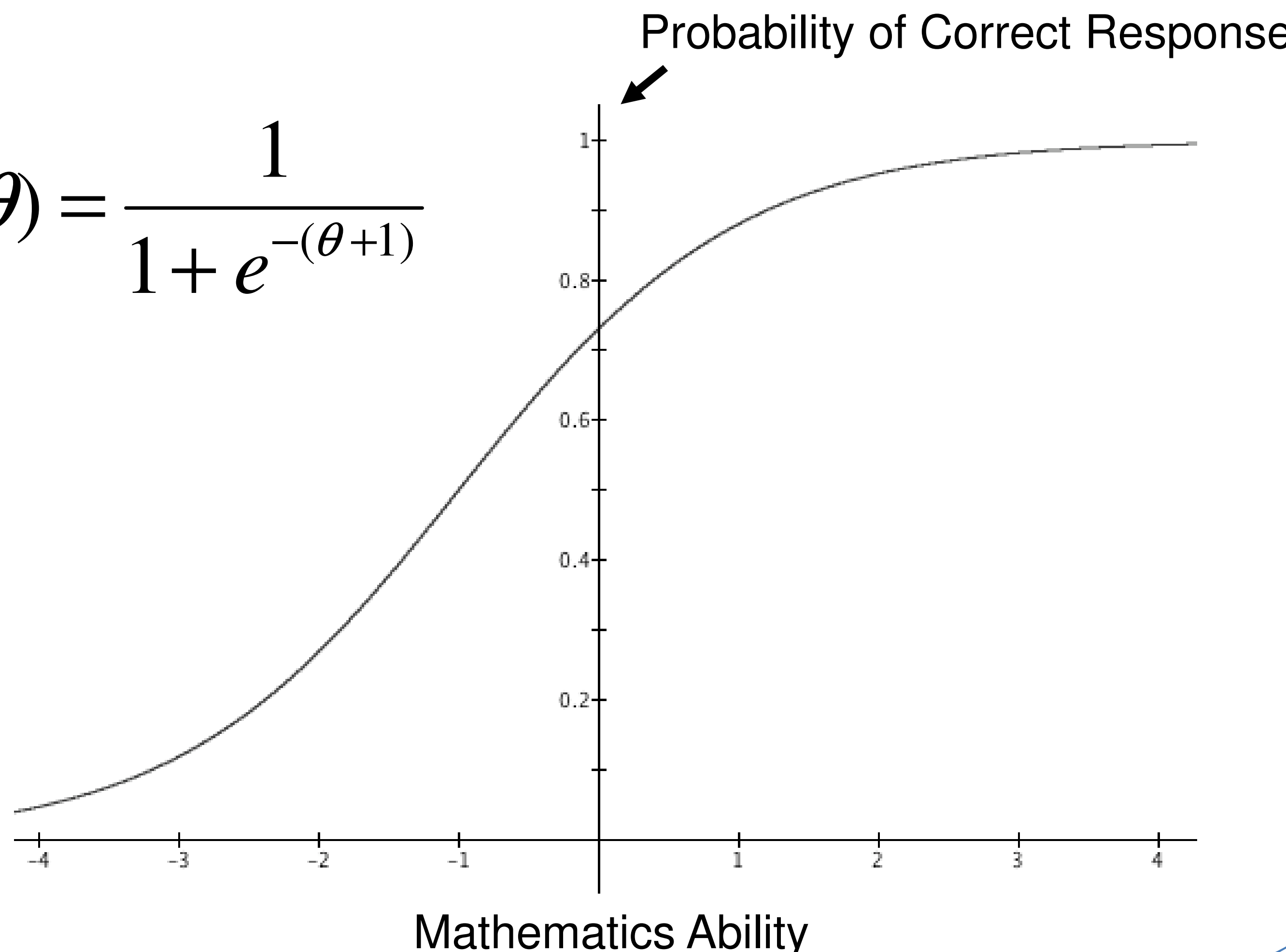
What is IRT?

- Family of psychometric models used to construct tests and analyze test data (e.g., SAT, GRE, NAEP)
- Theory based on individual questions (items) that make up a test
- Responses to items used to estimate latent variables (e.g., a person's ability in a given domain)
- Unidimensional scaling:



One-Parameter IRT Model
(Rasch Model)

$$P(\theta) = \frac{1}{1 + e^{-(\theta + 1)}}$$



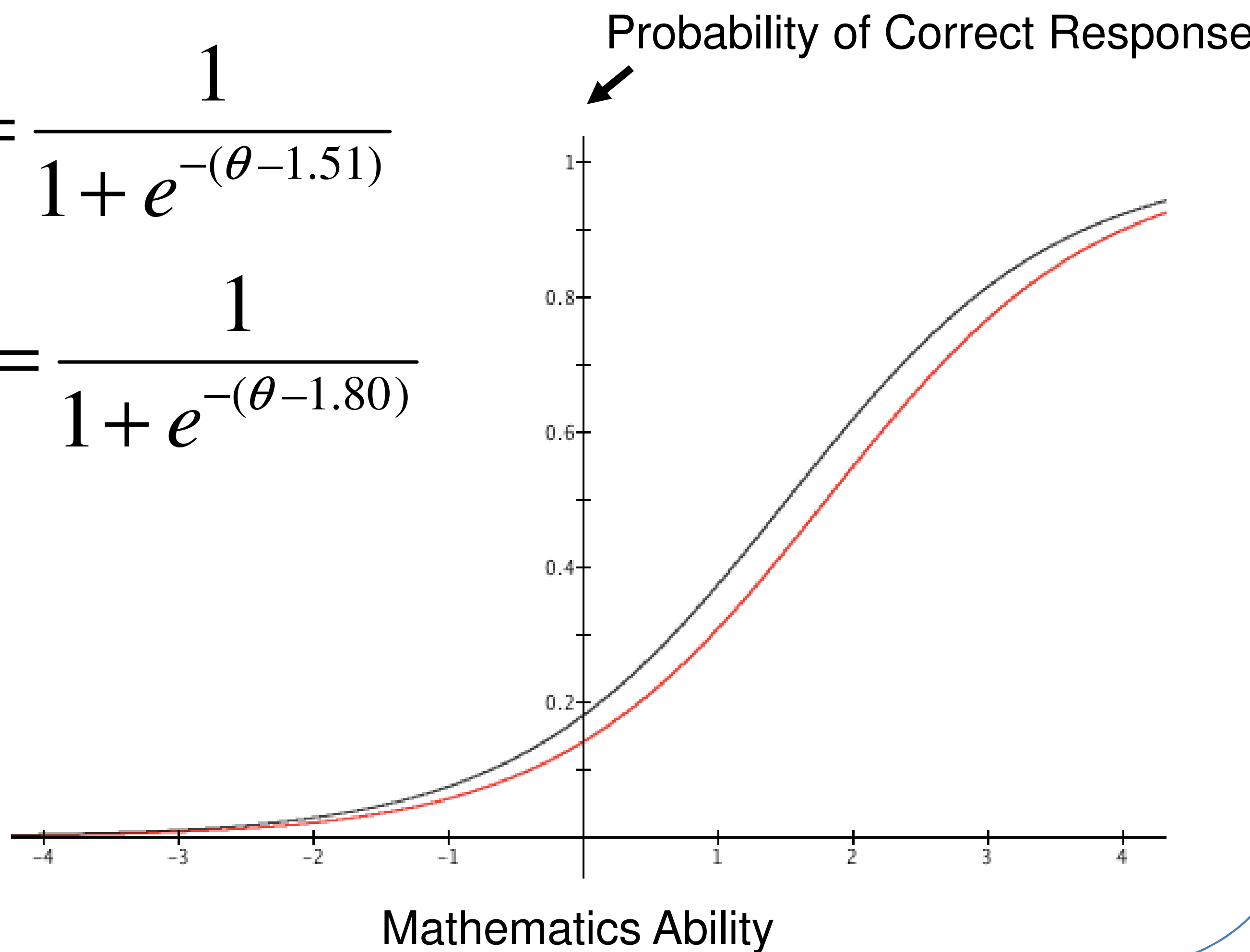
Study 1: Applying the Mixture-Rasch Model

- Izsák, Orrill, Cohen, & Brown (2010)
- Administered test to 201 middle school teachers in 4 states (convenience sample)
- Combined scaling with classification
  - Latent groups correspond to homogeneities in response patterns
  - Does best Rasch (IRT) model fit occur when all teachers are treated as one group, as two groups, etc.
- Conducted interviews with 16 teachers

Separate Item Locations for Each Group

$$P_1(\theta) = \frac{1}{1 + e^{-(\theta - 1.51)}}$$

$$P_2(\theta) = \frac{1}{1 + e^{-(\theta - 1.80)}}$$



Two Group Solution

- 2 groups (102 in Group 1, 99 in Group 2)
- Group 1 contains higher proportion of teachers whose responses are consistent with reasoning about referent units appropriately
- About 1/2 of Group 1 and 1/5 of Group 2 responded with correct choice for subtraction on number line
- Most common incorrect response for both groups:

1/8 of 1/5

Study 2: Studying Teachers in Professional Development

- Izsák, Jacobson, de Araujo, & Orrill (2011)
- 40-hour course (3 hours per week)
- Urban district in the Southeast
- 13 teachers (Grades 5, 6, and 7) and one district person (separate sample)
- Facilitated by member of the research team
- Whole-class discussion/group work
- Emphasis on referent units and drawn models

Data

- Pre-test/post-test constructed from item pool
- Videotaped each class and pre/post interviews with 7 teachers
- Written work
  - Problem write-ups
  - Reflections

Pre-Test to Post-Test: Ability

	Pretest			Posttest		
	Ability	Group	Prob.	Ability	Group	Prob.
Keith	2.73	1	0.98	2.22	1	0.95
Will	0.36	1	0.92	0.76	2	0.67
Linda	Ğ 0.40	2	0.77	0.18	2	0.79
Walt	1.48	1	0.84	1.60	1	0.98
Rose	Ğ 0.31	2	1.00	0.16	2	0.91
Pascal	Ğ 0.21	2	1.00	0.89	1	0.93
Donna	0.22	2	0.70	1.60	1	0.99
Carrie	Ğ 0.52	2	0.87	0.29	2	0.99
Claire	1.77	1	0.98	2.02	1	0.98
Salihah	Ğ 0.86	2	1.00	Ğ 0.09	2	1.00
Mike	1.33	1	0.50	2.23	1	0.97
Sharlene	1.18	2	0.91	0.55	1	0.84
Joyce	Ğ 0.40	2	1.00	0.29	2	0.98
Diane	1.24	1	0.79	1.53	2	0.86

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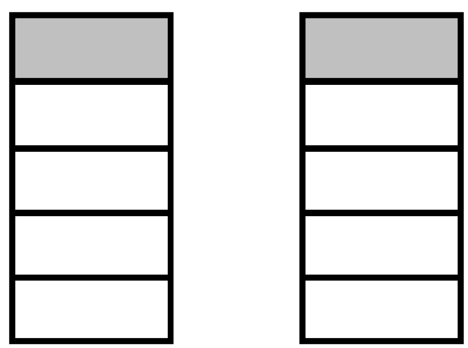
What is Behind Class Stability?

- Teachers in Class 1 evidenced 3 levels of units
- Teachers in Class 2 seemed constrained to 2 levels of units
- Example:
  - Share two candy bars equally among five people. How much of one candy bar does one person get?



Project 3: Diagnosing Teachers' Multiplicative Reasoning

- Fractions, Ratios, and Drawn Models
- Diagnostic Classification Models
- Select attributes identified as important in the research on students' and teachers' thinking
- Use attributes for multi-dimensional classification
- Confirmatory analysis

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Fractions Attributes

- **Referent Units**: Identifying units to which numbers refer
- **Partitioning**: Subdividing quantities into equal-sized parts
- **Iterating**: Interpreting A/B to mean A copies of 1/B
- **Appropriateness**: Recognizing situations that can be modeled by multiplication or division

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5/40 of 1

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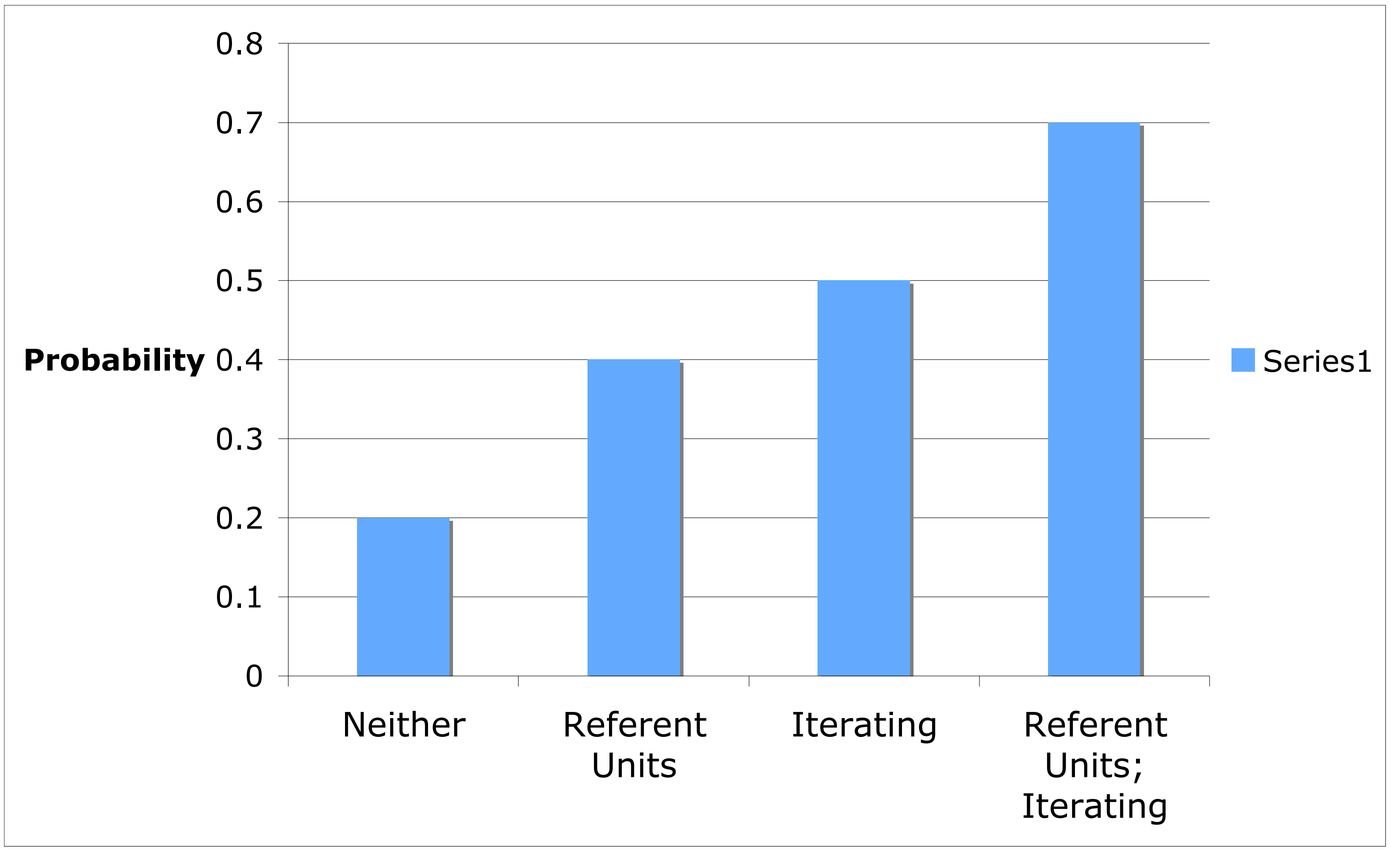
5/30 of 1

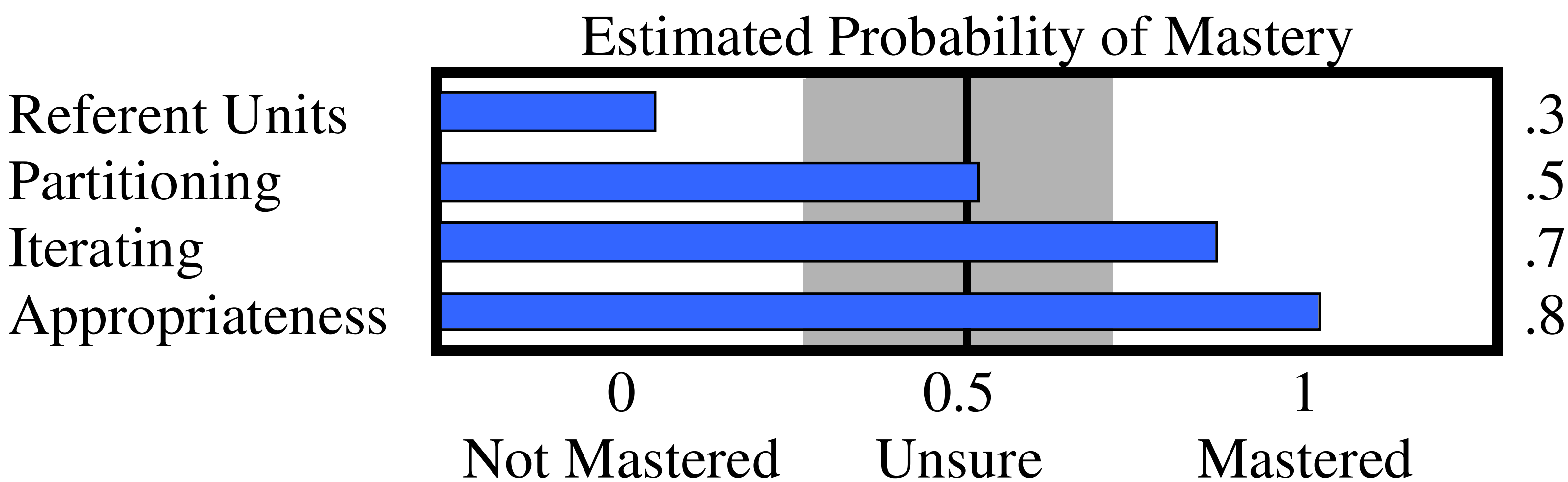
Probability of Item Response



Q Matrix

Item	Ref. Units	Part.	Iter.	Appr.
Item 1	1	1	0	0
Item 2	0	1	1	0
Item 3	1	0	0	0
Item 4	0	0	0	1
…				
Item j	q_{j1}	q_{j2}	q_{j3}	q_{j4}
…				

Mastery Profile



Learning About Teacher Knowledge Through Item Development

- Initial set of attributes
- Write items that measure one or more attributes
- Interview teachers to see if their reasoning is consistent with intended attributes
- Teachers have difficulty
  - Identifying appropriate referent units
  - Using knowledge of whole number multiplication as a resource for partitioning
  - Using iterating as a fundamental meaning for fractions

Conclusions

- Two vs. three levels of units helps explain why there are two groups of middle school teachers.
- Two vs. three levels of units could be an important focus for mathematics teacher education and professional development.
- There are many opportunities for innovative combinations of psychometric models and mathematics (and STEM) education research.