Modeling Students' Mathematics

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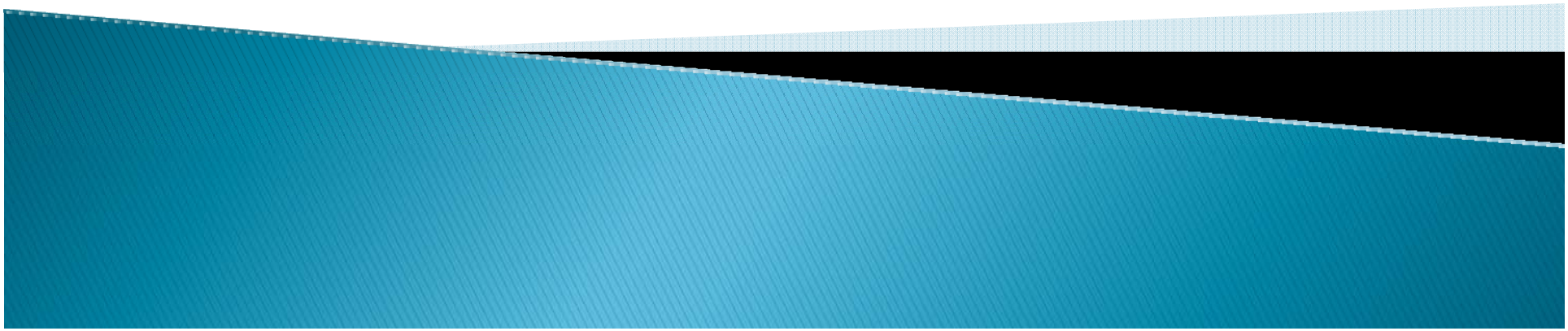
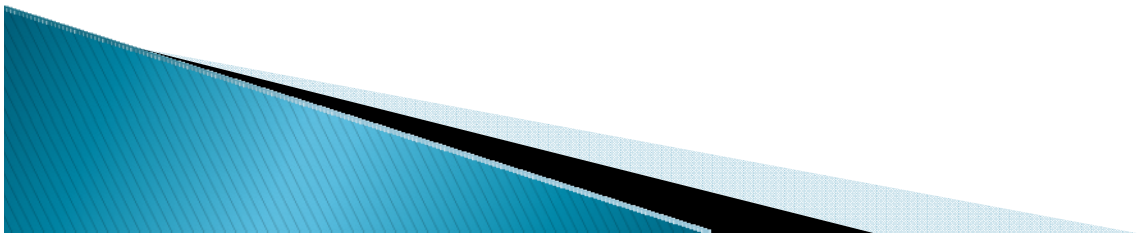


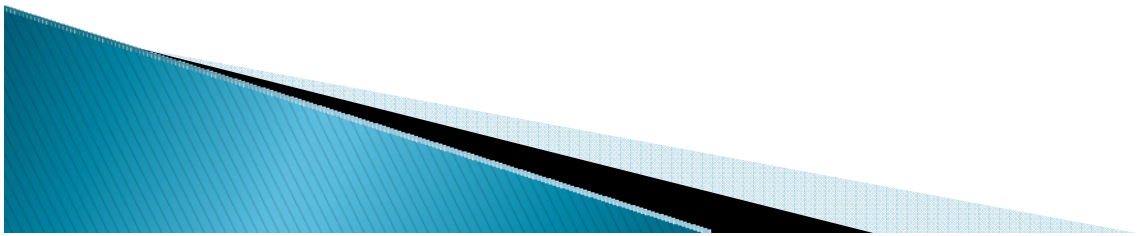
Starter Problem

- What does three-fifths mean?



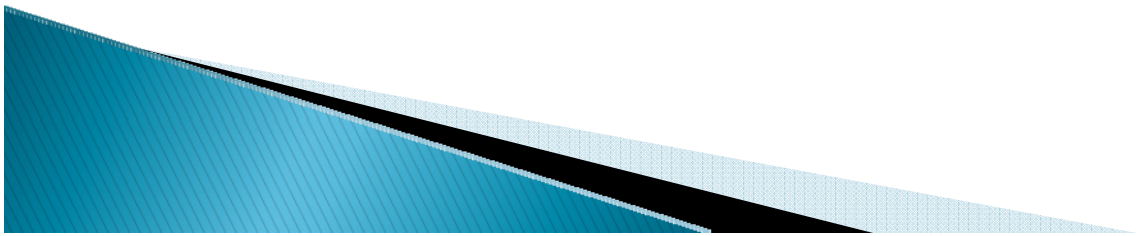
Common Core Standards

- 3.NF.1. Understand a fraction 1/*b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b.*
- 4.NF.4. Understand a fraction *a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product* $5 \times (1/4)$*, recording the conclusion by the equation* $5/4 = 5 \times (1/4)$.



Purpose

- “If, as teachers, we want to foster understanding, we will have a better chance of success once we have more reliable models of students’ conceptual structures, because it is precisely those structures upon which we hope to have some effect.”
  - von Glasersfeld & Steffe, 1991, p. 102



Understanding: The Role of Mental Operations

- The candy bar shown below is five times as big as your candy bar, draw your candy bar.

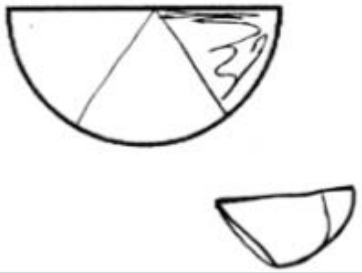
- What mental actions did you perform to solve the task?

Students' Available Mental Operations

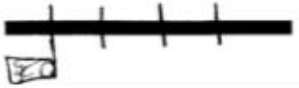
	Grade 6 (*N* = 66)		Grade 7 (*N* = 56)		Grade 8 (*N* = 58)	
Scheme/Operation	*M*	*SD*	*M*	*SD*	*M*	*SD*
Iterating	.86	.35				
Partitioning	.87	.33				
Partitive Unit Fraction Scheme	61	.49	.61	.49		
Splitting	.44	.50	.55	.50	.66	.48
Partitive Fraction Scheme	.17	.38	.25	.44		
Reversible Partitive Fraction Scheme			.13	.33	.19	.40
Iterative Fraction Scheme					.12	.33
Units Coordination					.26	.44

Splitting Tasks

(a) 4. The amount of pizza shown below is 3 times as big as your slice. Draw your slice.



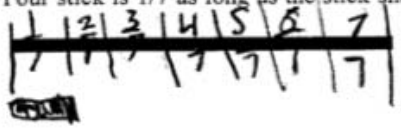
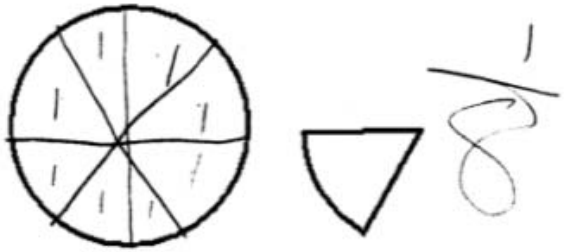
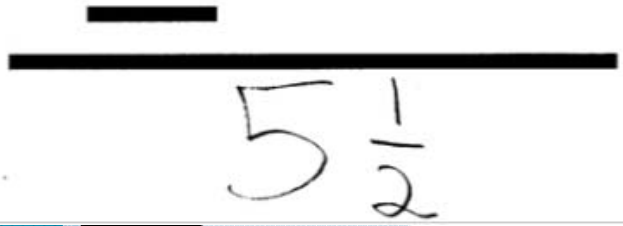
(b) 16. The stick shown below is 5 times as long as another stick. Draw the other stick.



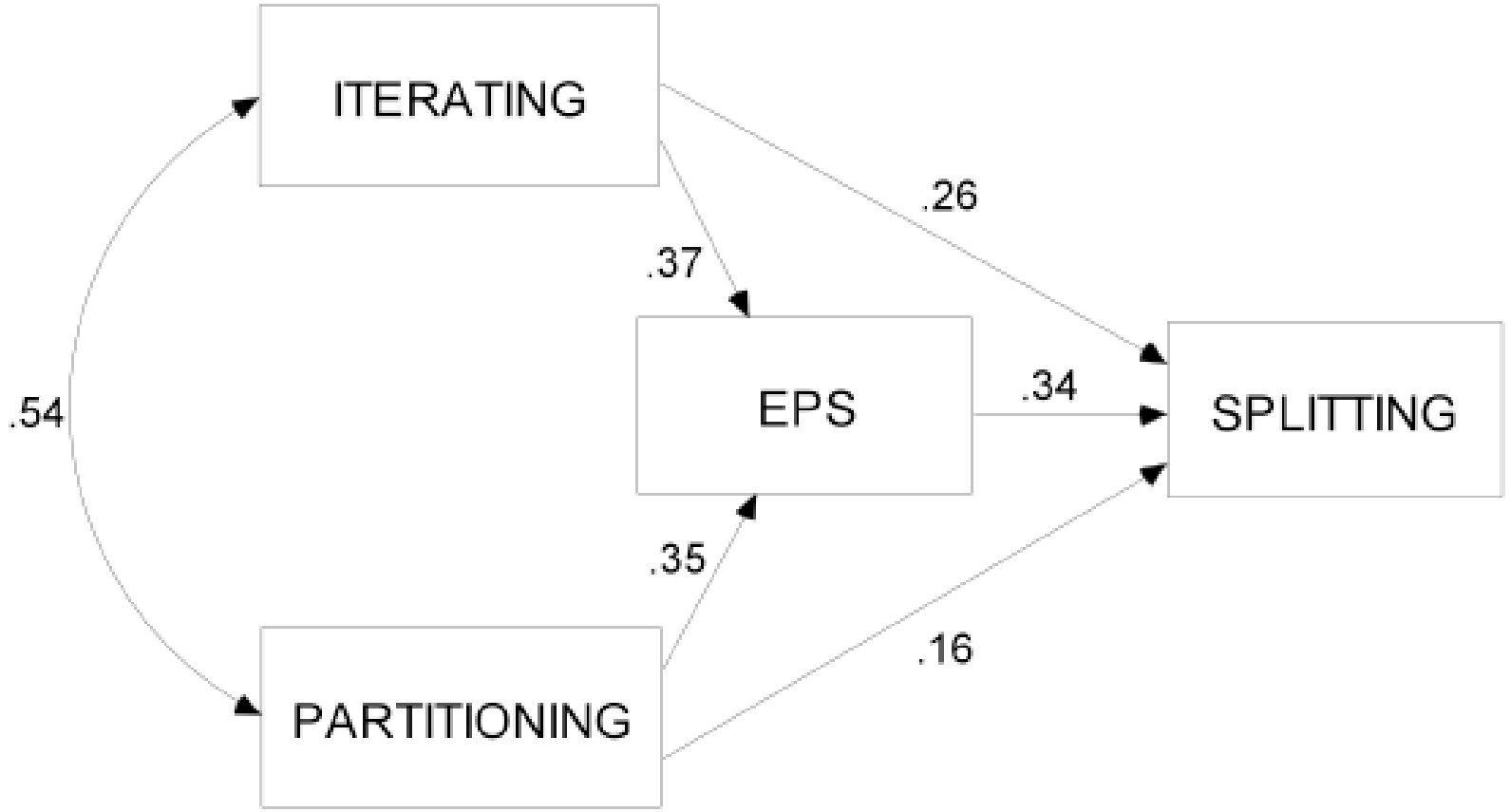
(c) 2. The stick shown below is 3 times as long as another stick. Draw the other stick.

(d) 9. The amount of pizza shown below is 6 times as big as your slice. Draw your slice.

PUFS/EPS Tasks

(a)	1. Your stick is 1/7 as long as the stick shown below. Draw your stick. 
(b)	6. What fraction is the smaller pie piece out of the whole pie? 
(c)	11. Your piece of pie is 1/5 as big as the piece shown below. Draw your piece of pie. 
(d)	4. What fraction is the smaller stick out of the longer stick? 

The Splitting Loope



Wilkins & Norton, JRME, 2011

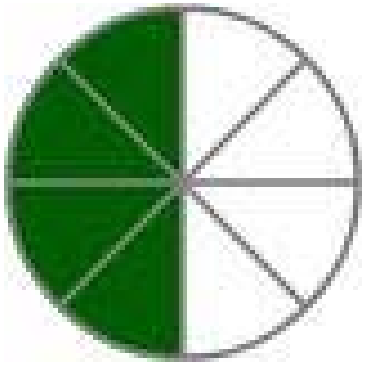
The Partitive Unit Fraction Scheme

- What fraction is the smaller stick out of the larger stick?

- Imagine what mental operations would be involved in generalizing this reasoning to *non-unit* proper fractions--the Partitive Fraction Scheme.

Isaac, 6th grade, April 26

- Andy: What is bigger, a fourth or an eighth?
- Isaac: an eighth
- Andy: Show me.
- Isaac: [draws circle with eight equal parts]

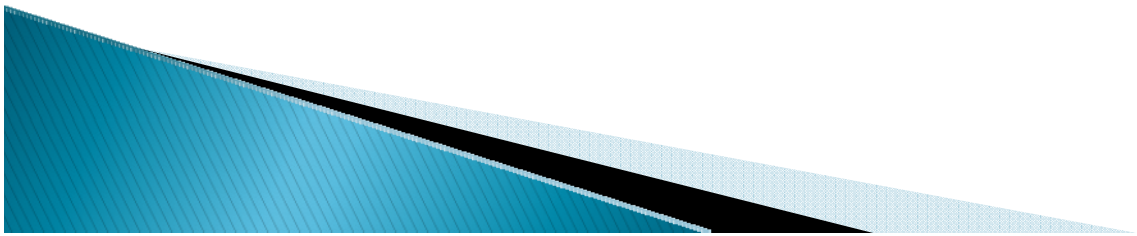


- Andy: If this was a pizza and you could have one fourth of the pizza, how many slices would you get?
- Isaac: Four
- Andy: What does it mean to be a fourth?
- Isaac: [long pause, as if confused]
- Andy: Does one fourth mean one out of four?
- Isaac: Sometimes.
- Andy: What else might one fourth mean?
- Isaac: Maybe a half.

Isaac & Kadyeisha, May 4

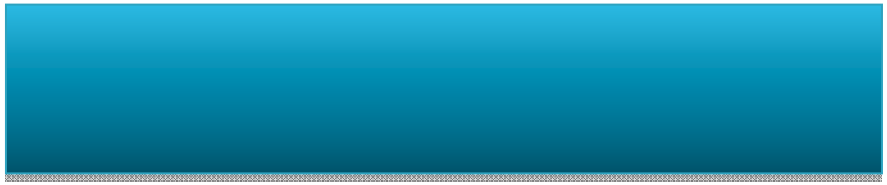
Textbooks

- “Throughout [the curricular development of fractions], which may take place over a few grade levels, the meaning of fractions seems to stay constant—part of a whole. As Thompson and Saldanha (2003) note, rarely do we see in the U.S. textbooks a treatment of non-unit fractions as a collection of unit fractions.”
  - Watanabe, 2007, pp. 55-56
- “In every county, mathematics textbooks exert a considerable influence on the teaching and learning of mathematics.”
  - Howson, 1995, pp. 5-6



Reversing the Partitive Fraction Scheme

- If the piece shown below is 3/5 of a candy bar, what does the whole look like?



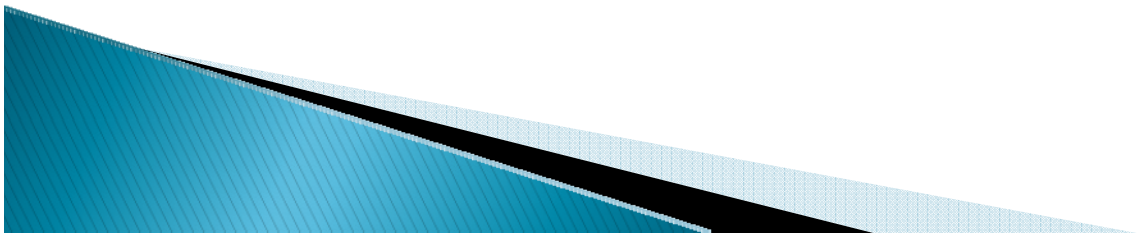
- What might Isaac say?

Isaac, May 12

- Andy: If this [green stick] is three-fifths of a whole candy bar, what does the whole candy bar look like?
- Isaac: [adds on four more bars]
- Andy: If this [green stick] is three-fifths, what does one-fifth look like?
- Isaac: [picks a red stick, one third the size of the green stick]
- Andy: …what does the whole look like?
- Isaac: [appends red stick to green stick] That. Two.
- Andy: [repeats question, upon request]
- Isaac: [puts five reds together] …four more on.

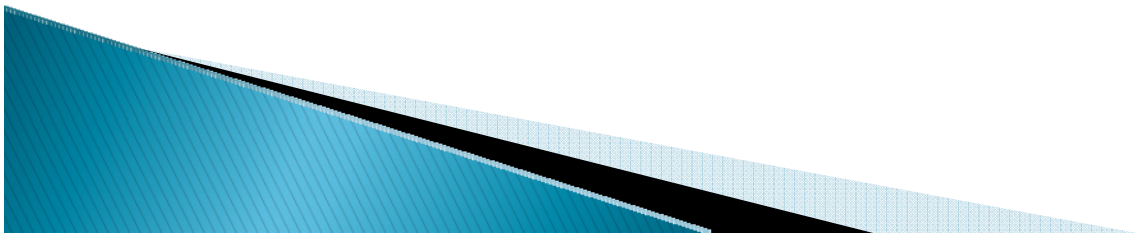
The Iterative Fraction Scheme

- "Fractions as numbers in their own right"
  - Hackenberg, 2007
- Again, what mental operations would be necessary to solve the previous task when generalized to *improper* fractions.

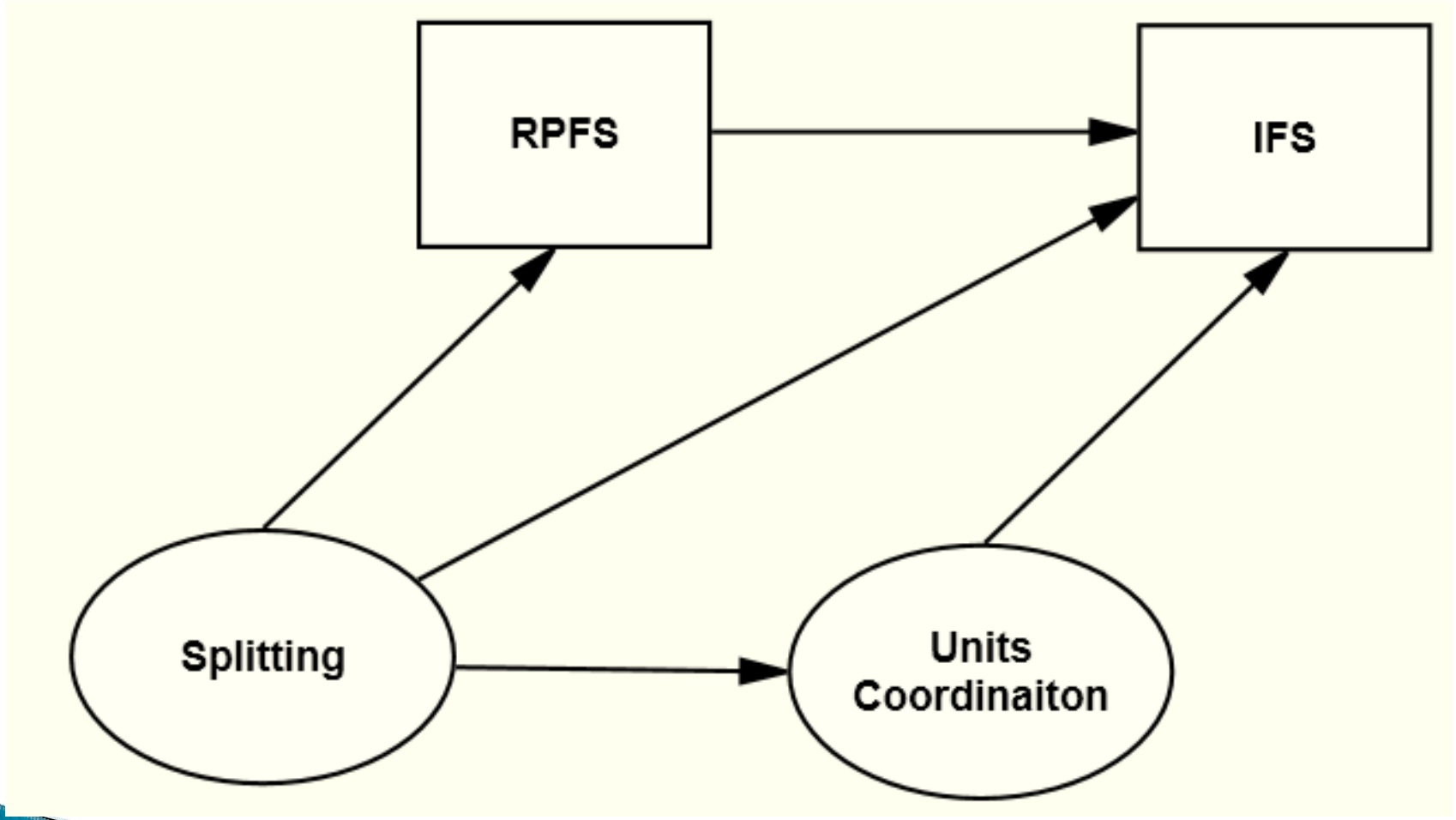


Three levels of units coordinating

- 24 as a unit of six units of 4 units.
- Seven-fifths as a unit of seven units of one-fifth, five of which produce the whole unit.

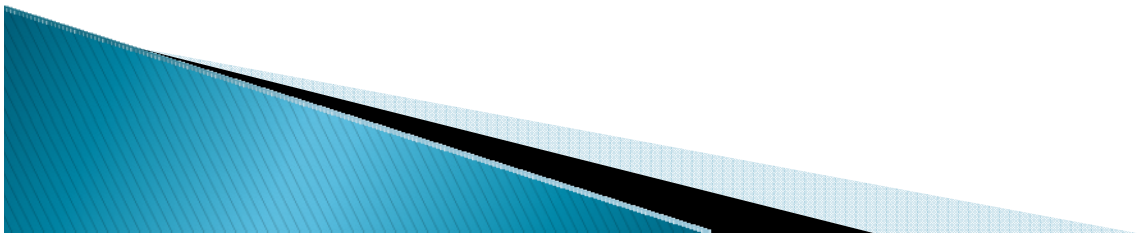


Constructing the Iterative Fraction Scheme



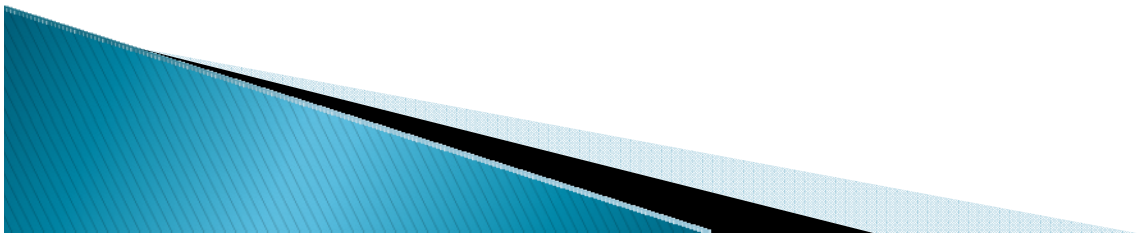
Isaac and Kadyeisha, May 17

- Andy: Imagine in your heads how much you would get if I said you could have three-fifths of that candy bar.
- Isaac: [uses fingers to span about one-third of the candy bar]
- Andy: Ok. And when you get three-fifths of it, what does that mean?
- Isaac: Oh! Wait! I thought you meant like… I thought you said *one* fifth!
- Isaac: [later] I mean, I thought you said one *third*.



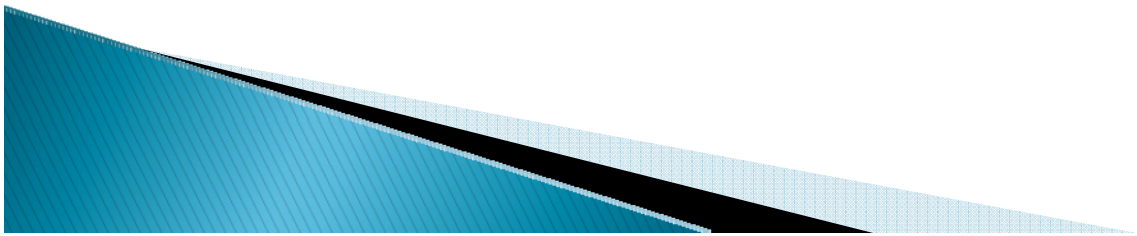
Understanding Fractions

- Andy: Think to yourselves for a second. If somebody asked you, ‘what does three fifths mean?’ and you had to explain it to them, just in words, think about what you would say to them.
- Kadyeisha: Three out of five
- Isaac: Yeah. Three out of five. Bigger than a half.



A Cognitive Core for Common Standards

- Common Core Standards place a much-needed emphasis on iterating unit fractions.
- In order for students to meet these standards, teachers need to understand the cognitive core with which students operate, and deliberately support students' constructions of new ways of operating
- Teachers, as model builders, must address the gap between cognitive core and common core.



Professional Development

- Cognitively Guided Instruction (CGI), Iterative Model Building (IMB), Studying Teachers' Evolving Perspectives (STEP), and Fractions Recovery projects
- Prediction Assessments
  - see Norton, McCloskey, & Hudson, JMTE, 2011