High-School Geometry: Traditions, Standards and the Preparation of Teachers

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I would like to use the time available to me make a few points about preparing secondary teachers to present geometry in manner consistent with the Common Core Standards. I will leave plenty of time for discussion.

Introduction. We know that good math teachers have a repertoire of skills and knowledge—much of it deeply mathematical—that is specialized for teaching. What role do college courses and professional development play in the acquisition of this professional repertoire? In particular, how should we engage future teachers in geometry? During the past decade at LSU, I have helped to design and develop of mathematics courses (both undergraduate and graduate) and professional development for secondary math teachers. As guiding principles, I have accepted ideas that I think many people agree on. Math teachers ought to: have a coherent understanding of the mathematics that they teach—and much beyond that; be able to articulate that understanding in many clear and precise ways; internalize many mathematical "structured event/representation complexes"; have the ability to make and test hypotheses about the contents of learners' minds and to react in appropriate ways; and much more. I am not sure, however, of the best ways to conduct a class to achieve these ends. I am concerned about the gap between the classroom where I teach and the classrooms in which my students teach, and fear that these may sometimes be parallel universes, side-by-side yet sharing no parts. I try to conduct classes that model the principles like those in the CCSSI Standards for Mathematical Practice. But there is too much I don't know about how the experiences I provide connect with the experiences my students eventually provide for their students, and I wish there were more research that would give me specific guidance about how to design courses. On the other hand, it is surely my responsibility to track the results of what I am doing into classrooms and make sure that the outcomes are as desired. Do I do this? No! I wish I could, but present, it's not humanly possible. Research will not relieve me responsibility to assure that what I teach and how I teach it achieves the desired outcomes, but research might provide the technology that would make it possible for me to do get the feedback I need.

Teacher Expertise. Many of the things that excellent teachers do in their classrooms share the qualities of expert performance that have been observed in other professions (e.g., medicine, writing, sports, music and mathematics). For example, when interacting with students, teachers manage their own cognitive resources, dividing them efficiently among many simultaneous tasks; they process complex information from numerous sources with extraordinary speed and draw on a vast stores of knowledge both about learners and about mathematics; they make effective decisions, often with little conscious deliberation, and they do all of this with apparent ease. Presently, LSU's NSF MSP Institute is supporting a team of psychologists (Drs. Sean Lane and Robert Matthews) who are studying teacher practice within the established psychological framework of "expert performance." This line of investigation was pioneered by K. Anders Ericsson, currently Professor of Psychology at Florida State University. Reference [1] contains a comprehensive review of research in this area.

One thing that may distinguish teaching from other professions is the opportunity for what is called "deliberate practice." This is work that practitioners undertake specifically to improve performance beyond its current levels; it is highly demanding, requiring intense focus and concentration; it entails goal-setting, preparation and most importantly feedback—preferably from an expert coach or mentor—as well as self-observation and self-reection. Moreover, it is sustained and repeated over long of periods of time. Professionals with the highest levels of expertise typically have engaged in thousands of hours of deliberate practice within their specialty. Yet, while it is easy to identify the kinds and occasions of deliberate in some professions, it is not clear what forms of deliberate practice teachers engage in. In fact, the question of deliberate practice in the teaching profession is a research question that has received attention in the expert performance literature. The LSU researchers have worked on this foundation and they have some findings and some hypotheses. Research based on student test data has shown repeatedly that teachers tend to improve during their first few years, but soon reach a plateau. This could be because test data are able to detect only some kinds kinds of improvement, or it could be because opportunities to develop high levels of skill are rare or are rarely undertaken.

Given that the entire undergraduate experience typically comprises about 3000 hours of active academic experience (including study) and that a typical teacher spends about 1000 hours in the classroom each year, it is clear that teaching expertise is not something we would expect to develop in a college course, nor even begin to see until a teacher is several years into his or her career. If this is the case, what should be the goals of a college course that is intended to prepare teachers? What kind of work should teachers in such a course do? If we cannot expect to create expertise, then perhaps we can prepare the ground for its development, assuring that teachers know what it looks like, aspire to develop it and appreciate the nature of the work that it requires.

"Mathematical Engineering."* But the challenge is not just to create expertise, it is also to create the conditions under which expertise can have a meaning and a purpose. No matter how skillful teachers are, their skill is only as valuable as the learning that it produces. Pointless material, after all, might be taught very skillfully. University mathematicians may engage in developing skillful math teachers, but they also should be engaged in perfecting the mathematical culture in which teachers and students are immersed. Mathematicians should help to shape mathematical knowledge so that it serves the purposes of education at all levels, from advanced study to the early elementary grades. They may strengthen the mathematical foundations that a great educational system demands, by creating documents that show that the curriculum is justified on intellectual grounds and that furnish a scholarly guide to its architectural plan. Euclid's *Elements* is the most ancient and most influential of all such documents. In modern times, Moise's book, *Elementary Geometry from an Advanced Standpoint* served these goals very well. History shows that scholarly work at an advanced level can profoundly influence the intellectual culture that is shared by people at all levels. This is evident not only in academics, but in other intellectual spheres as well, such as law and religion. Knowledge is socially distributed; people in different roles acquire knowledge of different kinds. The coordination and control of such socially distributed knowledge is dependent upon deep intellectual work preserved in written records.

Traditions in geometry. A pedagogical tradition is apparent in many of the most commonly used high school geometry textbooks; see page 4, below. They share terminology, selection and arrangement of topics, conceptual flow, kinds of exercises, etc. Many mathematicians find the presentation to be conceptually shallow and procedure-oriented. The connections among the themes are obscure, though as we will see, there are very deep connections. Speaking personally, my greatest concern about these books is not so much with what they contain as with the opportunities for depth and coherence *at no great cost* which they repeatedly miss. Clearly, these books serve a purpose, and if they did not serve it well, they would not be so widely used. But that purpose, whatever it is, cannot possibly be incompatible with a few thoughtful pages. I can point to the

* This is a term invented by Hung-Hsi Wu that refers to "the customization of basic mathematical principles to meet the needs of teachers and students." See [2].

treatment of the volume and surface area of a sphere as a place where some beautiful, classical mathematics could be presented, but where the opportunity is passed over.

As evidence of the pervasiveness of this paradigm, when asked to devise a normative curriculum for the state, Louisiana teachers came up with the following:

The units of the *Louisiana Comprehensive Curriculum in Geometry*, LA Dept. of Ed., 2008:

1. Geometric Patterns and Reasoning
2. Reasoning and Proof
3. Parallel and Perpendicular Relationships
4. Triangles and Quadrilaterals
5. Similarity and Trigonometry
6. Area, Polyhedra, Surface Area, and Volume
7. Circles and Spheres
8. Transformations

The curriculum in all the example above—and that in the books whose contents is exhibited on age 4—is for the most part just Euclid's *Elements*. I would not say merely "Euclidean Geometry," for that might refer to the study of the mathematical object that Euclid's postulates pick out, and this might be undertaken in a number of ways other than the way of the *Elements*, e.g., by studying a model such as $\mathbb{R}^2$ equipped with the Euclidean distance, etc. I mean, all these books clearly pattern the presentation of geometry directly on the ancient Greek canon (see page 5), though there are minor re-arrangements, dilutions and omissions.

The most significant departure from the *Elements* is the incorporation of a **metric** (measurement of distances by numbers), which for good reasons is fundamental in all contemporary high-school texts. This was not in the *Elements*. The employment of a universal unit of measurement was an innovation of Descartes' *Geometry*. The treatment of measurement in modern high school texts is strongly influenced by Birkhoff's axiomatization, which includes the Postulates of Line and Angle Measure (now called the Ruler and Protractor Postulates).

It is interesting that the order of presentation of Euclid has been modified so that parallel lines, and many propositions that depend on the Parallel Postulate, appear before congruence and congruent triangles are mentioned. To me, this seems to be a peculiar decision, but perhaps there is a pedagogical rationale. It may be easier to engage naive learners in reasoning about parallel lines and transversals than in triangle congruence. It would be interesting to pinpoint the origin of this plan.

In the texts I have listed, *transformations* make an awkward and unmotivated appearance. *Coordinates* are sprinkled throughout, and are always treated as though they were simply an alternative way to depict the geometry. In Moise's book—and in the geometry courses I present to teachers—coordinates are treated as a kind of geometric construction, and the properties of coordinates are explained. (Why is the graph of a linear function a straight line?)

Contents of some popular high-school geometry texts. Chapter titles are followed by *my comments on the topics treated.* (See the last page of these notes for the contents of a CCSS geometry course.)

Larson, R., Boswell, L., Stiff, L. (2004). **McDougal Littell Geometry.** Houghton Mifflin.

1. Basics of Geometry. *Terminology. Incidence Axioms. Ruler and Protractor Postulates.*
2. Reasoning and Proof. *Implication, syllogism.*
3. Perpendicular and Parallel Lines. *Alternate interior angles; Euclid I: 27–30.*
4. Congruent Triangles. *SSS, SAS, ASA, AAS. Isosceles $\triangle$s. Euclid I: 4–8 and I: 26.*
5. Properties of Triangles. *Medians, altitudes, etc. Inequalities; Euclid I: 16–21.*
6. Quadrilaterals. *Polygons. Parallelograms; Euclid I: 33, 34.*
7. Transformations. *Definitions and illustrations.*
8. Similarity. *Ratio and proportion. Similar triangles; Euclid VI: 2–10.*
9. Right Triangles and Trigonometry. *Pythagorean Theorem. Trig ratios.*
10. Circles. *Tangents, inscribed angles; Euclid III.*
11. Area of Polygons and Circles.
12. Surface Area and Volume.

Bass, L., Charles, R., Johnson, A., Kennedy, D. (2004). **Prentice Hall Geometry.** Pearson.

1. Tools of Geometry. *Terminology. Incidence Axioms. Ruler and Protractor Postulates.*
2. Reasoning and Proof. *Implication, syllogism.*
3. Parallel and Perpendicular Lines. *Alternate interior angles; Euclid I: 27–30.*
4. Congruent Triangles. *SSS, SAS, ASA, AAS. Isosceles $\triangle$s. Euclid I: 4–8 and I: 26.*
5. Relationships within Triangles. *Medians, altitudes, etc. Inequalities; Euclid I: 16–21.*
6. Quadrilaterals. *Polygons. Parallelograms; Euclid I: 33, 34.*
7. Area. *Of parallelograms and triangles; Euclid I: 35–41. Pythagorean Theorem.*
8. Similarity. *Ratio and proportion. Similar triangles; Euclid VI: 2–10.*
9. Right Triangle Trigonometry. *Trig ratios.*
10. Surface Area and Volume.
11. Circles. *Tangents, inscribed angles; Euclid III.*
12. Transformations. *Definitions and illustrations. Symmetry.*

Boyd, C., Cummins, J., Malloy, C., Carter, J., Flores, A. (2005). **Glencoe Geometry.** McGraw Hill.

1. Points, Lines, Planes, and Angles. *Terminology. Incidence Axioms. Ruler and Protractor Postulates.*
2. Reasoning and Proof. *Implication, syllogism.*
3. Parallel and Perpendicular Lines. *Alternate interior angles; Euclid I: 27–30.*
4. Congruent Triangles. *SSS, SAS, ASA, AAS. Isosceles $\triangle$s. Euclid I: 4–8 and I: 26.*
5. Relationships in Triangles. *Medians, altitudes, etc. Inequalities; Euclid I: 16–21.*
6. Proportions and Similarity. *Ratio and proportion. Similar triangles; Euclid VI: 2–10.*
7. Right Triangles and Trigonometry. *Pythagorean Theorem. Trig ratios.*
8. Quadrilaterals. *Polygons. Parallelograms; Euclid I: 33, 34.*
9. Transformations. *Definitions and illustrations. Matrices.*
10. Circles. *Tangents, inscribed angles; Euclid III.*
11. Area of Polygons and Circles. *Of parallelograms and triangles; Euclid I: 35–41.*
12. Surface Area.
13. Volume.

A map of the relevant parts of Euclid's *Elements*.

Book I

Triangle congruence; angles; inequalities in triangles.

I: 4	SAS
I: 5, 6	Isosceles $\triangle$s
I: 7, 8	SSS
I: 9–12	Constructions: bisecting angles and segments, drawing perpendiculars
I: 13–15	Supplements, vertical angles
I: 16–21	Inequalities for angles and sides if a $\triangle$
I: 22, 23	Constructions: copy a triangle, copy an angle
I: 24, 25	Inequalities
I: 26	ASA, AAS

Parallel lines.

I: 27–29	Transversals
I: 30	Two lines parallel to a third are parallel
I: 31	Construction: parallel to ℓ through P
I: 32	Sum of the angles in a $\triangle$ is $180°$

Parallelograms.

I: 33–34	Conditions equivalent to being a parallelogram

Area by dissection.

I: 35, 36	Area of a parallelogram
I: 37-40	Area of triangles
I: 41, 43	More on area
I: 42, 44–46	Constructions

Pythagorean Theorem.

I: 46	$\cdots$
I: 47	Converse

Book III

Circles: center and chords (1–15).

Tangents (16–19).

Inscribed angles, segments, lines intersecting circles, etc. (20–37).

Book V

Ratio and proportion for magnitudes (not numbers).

Book VI

Similarity, based on Book V and theory of area from Book I.

Teaching College Geometry to Preservice Secondary Teachers at LSU

- In Louisiana, secondary math teacher certification requires a college course in geometry. At LSU, most teachers meet this requirement with Math 4005: Geometry.
- In the 1990s, most instructors taught from Marvin Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History.*
- In the 2000s, I began experimenting with other sources and structures. My course was intended to make the existing culture of school geometry work, not alter it. Accordingly, Euclid's *Elements* exerted a powerful force on the design. The influence has grown greater over time.
- The fact that Euclid has virtually no mathematical prerequisites makes it accessible. Several people who have made serious attempts to teach from the *Elements* have observed this. The reactions of most of my students to the *Elements* have positive; they get deeply involved and they appreciate the logical discipline.
- The author of a book on the history of the geometry curriculum writes, "[T]he *Elements* dictates a certain order that was motivated entirely by logic and economy, with no concern for pedagogical considerations." This nonsense. The quotation attributes to Euclid the motivations of the mathematicians who were searching for foundations in the late 19^{th} and early 20^{th} century. I can argue—but not here—that the *Elements* is as as profound pedagogically as mathematically. (Of course, this is not to suggest that it would be a good book to use in a high school geometry class. A textbook does not have to be good for *all* learners to be good pedagogically.)
- It is also nonsense to condemn Euclid for logical shortcomings or lack of rigor, for this is also to judge Euclid by professional mathematical standards of the present day. The *Elements* is not *logically rigorous* in the modern "hard-nosed" sense. But it is *logically disciplined* in a sense that could be emulated usefully in school curricula today.
- Since my course was designed to support—not modify—existing curricula, transformations were treated toward the end. (I was never able to provide what I thought was a satisfactory motivation, but this is a consequence of the tradition I chose to work within.)
- A fairly complete picture of the course, as I taught it in 2006, may be viewed at my web site [3]. See [4] for an example of a lesson—or, more precisely some materials to support a lesson—that I think worked very well in the context of this course to promote the Standards for Practice. This is a small part of a much more elaborate lesson developed and studied by Bruce and Marybeth Olberding (now at New Mexico State). Reference [5] contains a report on their work.
- Scott Baldridge also teaches Math 4005. He makes extensive use of Parker and Baldridge, *Elementary Geometry for Teachers*. This book presents a distillation of the geometry that in in the Singapore mathematics curriculum, and systematically develops skill in solving meaningful, complex problems (especially where measures are involved), constructing arguments and proving theorems. Scott has said (to me) that his book is essentially presentation of the parts of Euclid outlined on the previous page, with measurement integrated throughout and used to simplify wherever possible.
- The Common Core State Standards envision a high school geometry that is builds on an informal understanding of isometries and dilations acquired in middle school. The high school curriculum consolidates this understanding and builds upon it. Transformations, which play an incidental role in the most widely used geometry books today, become a pillar of geometry within the curriculum of the CCS Standards.
- **In light of this, I anticipate a major redesign of Math 4005!**

Conclusion.

Euclid's *Elements* has shaped the existing high school geometry curriculum. This is not an accident of history. The "mathematical engineering" done by Hilbert, Birkhoff, Moise and others took Euclid as a model, and their work has helped to preserve the Euclidean paradigm. The Common Core State Standards reorient geometry. A treatment of geometry from this new perspective can be found in Klein's book, *Elementary Mathematics from an Advanced Standpoint: Geometry*, and there are many more recent discussions of transformational geometry, e.g., Barker & Howe, Hartshorne, Libeskind, Yaglom. Though all of these are accessible, they study transformations only *after* developing a lot of basic Euclidean geometry. Hung-Hsi Wu is preparing a set of notes that show how to develop geometry in the order called for in the Common Core Standards in a logically disciplined manner . (Klein's book also contains a discussion of how this might be done.)

The Common Core plan for teaching geometry needs the kind of deep, disciplined, coherent and self-contained textual foundation that Euclid and his modern mathematical interpreters provided for the plan that we currently implement.

References

[1] Ericsson, K. A., Charness, N., Feltovich, P. J. & Hoffman, R. R. (editors). (2006). *The Cambridge Handbook of Expertise and Expert Performance* (Cambridge Handbooks in Psychology). Cambridge University Press.

[2] `http://math.berkeley.edu/~wu/ICMtalk.pdf`

[3] `https://www.math.lsu.edu/ madden/M4005s2006/`

[4] `https://www.math.lsu.edu/~madden/M4005s2006/SeaIsland/`

[5] `https://www.math.lsu.edu/~madden/Sec_Math_Site/Olberdings.htm`

Common Core Traditional Pathway: Geometry (from CCSS Appendix A.)

Units

1. Congruence, Proof, and Constructions
    - Experiment with transformations in the plane.
    - Understand congruence in terms of rigid motions.
    - Prove geometric theorems.
    - Make geometric constructions.
2. Similarity, Proof, and Trigonometry
    - Understand similarity in terms of similarity transformations.
    - Prove theorems involving similarity.
    - Define trigonometric ratios and solve problems involving right triangles.
    - Apply geometric concepts in modeling situations.
    - Apply trigonometry to general triangles.
3. Extending to Three Dimensions
    - Explain volume formulas and use them to solve problems.
    - Visualize the relation between two-dimensional and three-dimensional objects.
    - Apply geometric concepts in modeling situations.
4. Connecting Algebra and Geometry through Coordinates
    - Use coordinates to prove simple geometric theorems algebraically.
    - Translate between the geometric description and the equation for a conic section.
5. Circles With and Without Coordinates
    - Understand and apply theorems about circles.
    - Find arc lengths and areas of sectors of circles.
    - Translate between the geometric description and the equation for a conic section.
    - Use coordinates to prove simple geometric theorem algebraically.
    - Apply geometric concepts in modeling situations.
6. Applications of Probability
    - Understand independence and conditional probability and use them to interpret data.
    - Use the rules of probability to compute probabilities of compound events in a uniform probability model.
    - Use probability to evaluate outcomes of decisions.

The CCSSI Standards for Mathematical Practice

1. *Make sense of problems and persevere in solving them.*
2. *Reason abstractly and quantitatively.*
3. *Construct viable arguments and critique the reasoning of others.*
4. *Model with mathematics.*
5. *Use appropriate tools strategically.*
6. *Attend to precision.*
7. *Look for and make use of structure.*
8. *Look for and express regularity in repeated reasoning.*