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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes Email/Phone: SRHERMES@BRANDEIS.EDU

Speaker's Name: Idun Reiten

Talk Title: Tau-Tilting Theory

Date: 10 / 29 / 12 Time: 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Tilting theory, path algebras, representation theory, cluster category, cluster tilting

Please summarize the lecture in 5 or fewer sentances: The speaker reviewed the theory of tilting modules and cluster tilting objects. The speaker generalized these notions to the notion of tau tilting.

CHECK LIST

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Tau-Tilting Theory I

Idun Reiten — October 29, 2012

Joint with Adachi, Iyama.

1 Tilting Theory

Definitions/Notation: Let Λ be a finite dimensional k-algebra (say $k = \mathbb{C}$). Suppose $\Lambda = P_1 \oplus \cdots \oplus P_N$ with P_i indecomposable and $P_i \neq P_j$ for $i \neq j$ (i.e. Λ is *basic*). Let $\mathsf{mod}\text{-}\Lambda$ be finite dimensional Λ-modules.

Definition 1.1. T in $\mathsf{mod}\text{-}\Lambda$ is a *partial tilting module* if $\operatorname{pd} T \leq 1$ and $\operatorname{Ext}^1(T,T) = 0$. It is *tilting* if also $|T| = n = |\Lambda|$ (here $|M|$ denotes the number of indirect summands). It is *almost complete tilting* if also $|T| = n - 1$.

Example 1.2. $\Lambda = P_1 \oplus \cdots \oplus P_n$ itself is a tilting module. $P_1 \oplus \cdots \oplus P_{n-1}$ is almost complete tilting module.

Basic Properties:

1. Any partial tilting module can be extended to a tilting module.
2. An almost complete tilting module can be extended to a tilting module in exactly one xor two different ways to a titling module.

Example 1.3. Let Q be the quiver $1 \to 2 \to 3$, so $kQ = P_1 \oplus P_2 \oplus P_3$. If $T = P_2 \oplus P_3$, there are no extensions of T other than kQ.

Problem: Find a more general class of modules where we always have 2 complements. This leads us to τ-tilting theory.

2 Cluster Categories [BMRRT] and Support Tilting Modules [I-Thomas, Ringel]

Let Q be a finite acyclic quiver (i.e. no oriented cycles) so that kQ is finite dimensional k-algebra. Then the *cluster category* $\mathcal{C}_Q := \mathcal{D}^b(kQ)/F$ where $F = \tau^{-}[1]$.

Example 2.1. Let Q be the quiver $1 \to 2 \to 3$. The Auslander-Reiten quiver is:

$$\begin{tikzcd}
& & P_1 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] & & \bullet \arrow[ll, dotted] \arrow[dr] & \\
P_3 \arrow[ur] & & X \arrow[ll, dotted] \arrow[ur] & & \bullet \arrow[ll, dotted]
\end{tikzcd}$$

Have almost split sequences $0 \to P_3 \to P_2 \to X \to 0$ etc. and τ sends X to P_3 (similarly for other almost split sequences). The derived category is:

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & & P_3[1] \arrow[ll, dotted] \arrow[dr] & & X[1] \arrow[ll, dotted] \arrow[dr] & & \bullet \arrow[ll, dotted] \\
\cdots & P_2 \arrow[ur] \arrow[dr] & & \bullet \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & \bullet \arrow[ll, dotted] \arrow[ur] & \cdots \\
P_3 \arrow[ur] & & X \arrow[ll, dotted] \arrow[ur] & & \bullet \arrow[ll, dotted] \arrow[ur] & & P[1] \arrow[ll, dotted] \arrow[ur] & &
\end{tikzcd}$$

Get an extension of τ to the derived category. So $F = \tau^{-1}[1]$ sends for example P_3 to $X[1]$. So in $\mathcal{C}_Q$ we identify P_3 with $X[1]$ and so on.

The new indecomposables (other than the ones from $\mathsf{mod}\text{-}kQ$) are $P_1[1], P_2[1], P_3[1]$. In general we get the original modules and the shifts of the projectives.

The motivation of this cluster category was to categorify the acyclic cluster algebras.

Definition 2.2. T in $\mathcal{C}_Q$ is *cluster tilting object* if $\mathrm{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$ and T is maximal with this property. Equivalently, $|T| = n = \#(\text{vertices of } Q)$. (Here we can define Ext because $\mathcal{C}_Q$ is a triangulated category).

Theorem 2.3. An almost complete cluster tilting object has exactly two compliments.

Suppose T is cluster tilting object with $T = T_0 \oplus P[1]$ where T_0, P are kQ-modules and P is projective. We want to express the condition $\mathrm{Ext}_{\mathcal{C}_Q}(T, T) = 0$ in terms of the module category $\mathsf{mod}\text{-}kQ$. This is equivalent to the conditions

$$\mathrm{Ext}^1_{kQ}(T_0, T_0) = 0 \quad \text{and} \quad \mathrm{Hom}(P, T_0) = 0.$$

If $P = 0$, then we get exactly the tilting modules.

In general we get *support tilting modules*. (i.e. $\mathrm{Ext}^1(M, M) = 0$ and it is tilting modulo the summands that are not in the support.)

Example 2.4. Same quiver as before. Have composition series $P_3 = S_3$, $P_2 = (S_2, S_3)$, and $P_1 = (S_1, S_2, S_3)$. The support of $T = P_3 \oplus P_2$ is $2 \to 3$ so T is a support tilting module.

Consider $P_2 \oplus P_1$. Then the support is all of Q but it is not tilting. Hence not support tilting.

In this way we get a one to one correspondence between support tilting modules and cluster tilting objects. The theorem about having two complements translates to support tilting.

3 Cluster Tilted Algebras

Definition 3.1. The algebras $\Gamma = \mathrm{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}}$ where T is a cluster tilting object are called *cluster tilted.*

Example 3.2. $T = kQ$ is a cluster tilting object, so $kQ = \mathrm{End}_{\mathcal{C}_Q}(T)$ is cluster tilted.

We have an equivalence $\mathrm{Hom}_{\mathcal{C}_Q}(T, -) : \mathcal{C}_Q/\tau T \to \mathsf{mod}\text{-}\Gamma$ (cf. [BMR]). Again, we want to express the condition $\mathrm{Ext}_{\mathcal{C}_Q}(M, M) = 0$ in terms of $\mathsf{mod}\text{-}\Gamma$. It turns out that

$$\mathrm{Ext}^1_{\mathcal{C}_Q}(M, M) = 0 \quad \text{iff} \quad \mathrm{Hom}_\Gamma(\widetilde{M}, \tau\widetilde{M}) = 0$$

if M has no summand of the form τT; here $\widetilde{M}$ denotes the corresponding Γ-module (namely, $\mathrm{Hom}_{\mathcal{C}_Q}(T, M)$). This leads to the notion of τ-tilting theory.

Note for $\Gamma = kQ$, this is equivalent to $\mathrm{Ext}_\Gamma(\widetilde{M}, \widetilde{M}) = 0$.

Definition 3.3. Let Λ be a finite dimensional k-algbera with $|\Lambda| = n$. X in $\mathsf{mod}\text{-}\Lambda$ is *τ-rigid* if

$$\mathrm{Hom}_\Lambda(X, \tau X) = 0.$$

It is *τ-tilting* if also $|X| = n$ and *almost τ-tilting* if $|X| = n - 1$.

Remark 3.4. Tilting implies τ-tilting, and the two notions coincide for kQ.

Definition 3.5. A pair (M, P) where M, P are Λ-modules and P is projective is called *τ-rigid* if M is τ-rigid and $\mathrm{Hom}(P, M) = 0$. It is *support τ-tilting* if also $|M| + |P| = n$. It is *almost support τ-tilting* if $|M| + |P| = n - 1$.

Theorem 3.6. Let Γ be a cluster tilted algebra. Then any almost support τ-tilting module can be extended in exactly two ways to a support τ-tiling module.

Remark 3.7. Also can be done for 2-CY tilted algebras (Iyama-Yoshino; Keller-Reiten; Geiß-Leclerc-Schröer) in the same way.
In fact, the theorem holds for *any* finite dimensional algebra (as long as the analogous concepts make sense) (paper just appeared on ArXiv).

The proof is purely representation theoretic. So, we can prove the cluster tilting version as a corollary and translating the concepts into properties of $\mathcal{C}_Q$.