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NOTETAKER CHECKLIST FORM

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Name: Stephen Hermes Email/Phone: SRHERMES @ BRANDEIS.EDU

Speaker's Name: B. Keller

Talk Title: Nakajima Quiver Varieties ; Derived Categories

Date: 10 / 29 / 12 Time: 10 : 30 am / pm (circle one)

List 6-12 key words for the talk: Quiver Varieties, derived categories, monoidal categorifications, Nakajima category, Gorenstein Homological Algebra

Please summarize the lecture in 5 or fewer sentances: The speaker reviewed the motivation for studying quiver varieties in the context of monoidal categorification of cluster algebras. The speaker introduced Nakajima quiver varieties, and related their geometry to the derived category of a Dynkin quiver.

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Nakajima Quiver Varieties and Derived Categories

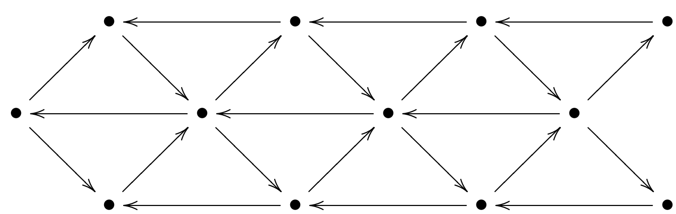
B. Keller October 29, 2012

Joint with Sarah Scherotzke. Builds on previous work with Hernandez-Leclerc (3-preprints [HL1], [HL2], [HL3]) and by Leclerc-Plamondon (preprint [LP]).

Motivation

Hope expressed by Nakajima (2009 preprint) that by using perverse sheaves on quiver varieties it should be possible to obtain monoidal categorifications (in the sense of [HL1]) of the cluster algebras associated with the T-system quivers $T_{Q,l}$. Here Q is an acyclic quiver, and $l \geq 1$ (the level).

Example 0.1. The quiver $T_{A_3,4}$:



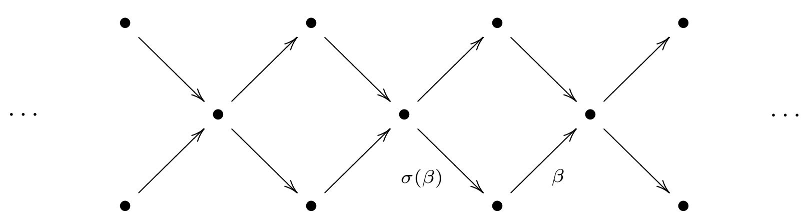
The existence of a monoidal categorification is very strong and in particular implies strong positivity of the corresponding cluster algebra.

Realized for $l = 1$ by Nakajima for Q *bipartite* (i.e. each vertex is either a source or a sink) to a certain extent. Generalized to the acyclic case by Kimura-Qin (to a certain extent). Using algebraic and combinatorial methods, realized by [HL1,HL3] for Q of type A, D fully.

Aim of the Talk: Get a better understanding of the quiver varieties associated with a Dynkin quiver Q and an arbitrary level $l \geq 1$. (Inspired by the work of [HL2,LP] using derived categories.)

1 Reminder on Repetitive Quivers and Happel's Theorem

Fix Q a Dynkin quiver, e.g. $Q : 1 \to 2 \to 3$. Denote by Q_0 its set of vertices and $\mathbb{Z}Q$ its repetitive quiver, e.g. for above Q the repetitive quiver $\mathbb{Z}Q$ is



which up to isomorphism is independent of the orientation of Q. In general, $\mathbb{Z}Q$ has vertices the pairs $(i,p) \in Q_0 \times \mathbb{Z}$ and and arrows given as follows: For each $\alpha : i \to j$ in Q we have two families of arrows:

1. $(\alpha, p) : (i,p) \to (j,p)$

2. $\sigma(\alpha, p) : (j, p-1) \to (i, p)$

Note $\sigma^2(\alpha, p) = (\alpha, p-1)$. We also have the translation automorphism $\tau : (i, p) \mapsto (i, p-1), (\alpha, p) \mapsto (\alpha, p-1)$. Fix a field k.

Definition 1.1. Define $k(\mathbb{Z}Q)$ to be the *mesh category* of $\mathbb{Z}Q$. The objects are the vertices of $\mathbb{Z}Q$. The morphisms are given by linear combinations of paths modulo the subspace generated by all ur_xv where u, v are paths and r_x is the mesh relation associated with the vertex $x \in Q_0$. Here

$$r_x = \sum_{\beta : y \to x} \sigma(\beta) \cdot \beta.$$

Example 1.2. For the quiver A_2 we have $k(\mathbb{Z}Q)$ is

$$\begin{tikzcd}
\cdots & & \bullet \arrow[dr] & & \bullet \arrow[dr, "\sigma(\beta)"'] & & \bullet \arrow[dr] & & \cdots \\
\cdots & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur, "\beta"'] & & \bullet & \cdots
\end{tikzcd}$$

where compositions of two consecutive arrows vanish in $k(\mathbb{Z}Q)$.

Definition 1.3. If kQ is the path algebra of Q, $\mathsf{mod}\text{-}kQ$ the finite dimensional right kQ-modules, we can form its bounded derived category $\mathcal{D}_Q := \mathcal{D}^b(\mathsf{mod}\text{-}kQ)$ of kQ-modules.

Theorem 1.4 (Happel '86)**.** We have a canonical equivalence of categories

$$H : k(\mathbb{Z}Q) \to \mathsf{ind}\ (\mathcal{D}_Q) = \{\text{indecomposable objects in } \mathcal{D}_Q\}$$

determined by $(0, i) \mapsto P_i = e_i kQ$.

2 Graded Affine Quiver Varieties

The *framed quiver* Q^{fr} as the quiver obtained from Q by adding certain frozen vertices. Namely, for each vertex i, add a new frozen vertex i' (usually drawn in boxes) and a new arrow $i \to i'$

Example 2.1. Let $Q : 1 \to 2$. Then Q^{fr} is

$$\begin{tikzcd}
2 \arrow[r] & {[2']} \\
1 \arrow[u] \arrow[r] & {[1']}
\end{tikzcd}$$

where the brackets [ ] indicate the vertex is frozen.

Define $\mathbb{Z}Q^{\text{fr}}$ as the repetitive quiver of Q^{fr} where all the new frozen vertices give rise to families of frozen vertices. That is, each (i', p) is frozen.

Example 2.2. For Q as in the previous example, $\mathbb{Z}Q^{\text{fr}}$ is

$$\begin{tikzcd}
\cdots & & \bullet \arrow[r] \arrow[dr] & {[\bullet]} \arrow[r] & \bullet \arrow[r] \arrow[dr] & {[\bullet]} \arrow[r] & \bullet \arrow[dr] & & \cdots \\
\cdots & \bullet \arrow[r] \arrow[ur] & {[\bullet]} \arrow[r] & \bullet \arrow[r] \arrow[ur] & {[\bullet]} \arrow[r] & \bullet \arrow[r] \arrow[ur] & {[\bullet]} \arrow[r] & \bullet & \cdots
\end{tikzcd}$$

Definition 2.3. Define $k(\mathbb{Z}Q^{\text{fr}})$ to be the mesh category of $\mathbb{Z}Q^{\text{fr}}$ with only the mesh relations associated to non-frozen vertices. The *regular (smooth) Nakajima category* $\mathcal{R}$ is this mesh category. The *singular Nakajima category* $\mathcal{S}$ is the full subcategory of $\mathcal{R}$ whose objects are the frozen vertices. Let $\mathcal{S}_0 = \{u = \sigma(x) : x \in \mathbb{Z}Q\}$ denote the set of objects in $\mathcal{S}$ (i.e. the set of frozen vertices).

An $\mathcal{S}$*-module* is a k-linear functor $M : \mathcal{S}^{\mathrm{op}} \to \mathsf{Mod}\ kQ$. Let $w : \mathcal{S}_0 \to \mathbb{N}$ be a *dimension vector*, i.e. a function with finite support. Associated to w we have a graded quiver variety $\mathcal{M}_0(w)$ consisting of the $\mathscr{S}$-modules M such that $M(u) = k^{w(u)}$ for all $u \in \mathcal{S}_0$. Note it is a closed subvariety of

$$\prod_{u_1,u_2\in\mathcal{S}_0} \left(k^{w(u_1)\times w(u_2)}\right)^{\dim \mathrm{Hom}_{\mathcal{S}}(u_1,u_2)}$$

Theorem 2.4 (LP)**.** This definition is equivalent to Nakajima's original definition. (Main ingredient of proof is an old theorem of Lustig.)

Remark 2.5. In order to make $\mathcal{M}_0(w)$ more explicit, we need a "minimal presentation" of $\mathcal{S}$. That is, a presentation of $\mathcal{S}$ by a quiver with relations (not just a subquiver of such). To do this, we need to compute Ext^1 and Ext^2 between the simple $\mathcal{S}$-modules $S_{\sigma(x)}$, for $x \in (\mathbb{Z}Q)_0$.

Theorem 2.6 (Theorem 1)**.** For any $p > 0$, we have

$$\mathrm{Ext}^p(S_{\sigma(x)}, S_{\sigma(y)}) = \mathrm{Hom}_{\mathcal{D}_Q}(H(x), H(y)[p])$$

where H is Happel's equivalence and $[p]$ is shift of complexes.

We can compute the right-hand side above explicitly in terms of the root system corresponding to Q.

3 Stratifications

Definition 3.1. Let $v : \mathcal{R}_0 - \mathcal{S}_0 \to \mathbb{N}$ and $w : \mathcal{S}_0 \to \mathbb{N}$ be dimension vectors, and set

$$G_v = \prod_{\substack{x \text{ non-} \\ \text{frozen}}} \mathrm{GL}(k^{v(x)}).$$

Define $\mathcal{M}(v, w)$ to be the smooth graded quiver variety

$$\{\mathcal{R}\text{-modules } M : M(x) = k^{v(x)}, M(\sigma(x)) = k^{w(x)}, \mathrm{Hom}(S_x, M) = 0 \text{ for each } x \in \mathbb{Z}Q_0\}/G_v.$$

Remark 3.2. Nakajima has shown

1. $\mathcal{M}(v, w)$ is quasi-projective
2. $\pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w)$ by $M \mapsto M|_{\mathcal{S}}$ is proper
3. $\mathcal{M}_0(w)$ is stratified by $\pi(\mathcal{M}^{\mathrm{reg}}(v, w))$

where $\mathcal{M}^{\mathrm{reg}}(v, w)$ is the regular G_v-orbits which is an open subvariety of $\mathcal{M}(v, w)$.

Theorem 3.3 (Theorem 2)**.** There is a canonical functor

$$\Phi : \mathsf{mod}\text{-}\Sigma \to \mathcal{D}_Q$$

such that $S_{\sigma(x)} \mapsto H(x)$. Moreover if $M_1, M_2 \in \mathcal{M}_0(w)$, then M_1 and M_2 lie in the same stratum iff $\Phi M_1 \cong \Phi M_2$ in $\mathcal{D}_Q$.

Remark 3.4. This is inspired by [HL2] and [LP] who obtain analogous results for certain w.

Theorem 3.5 (Theorem 3)**.** Consdier $\pi : \coprod \mathcal{M}(v, w) \to \mathcal{M}_0(w)$ and $M \in \mathcal{M}_0(w)$. Then the fibre under π of M is the Grassmannian of $\mathcal{D}_Q$-submodules of $\mathrm{Hom}(-, \Phi M) : \mathcal{D}_Q^{\mathrm{op}} \to \mathsf{mod}\text{-}k$.

Remark 3.6. This result is classical for M semisimple.

4 Gorenstein Homological Algebra

Let $\mathsf{gpr}(\mathcal{S})$ be the category of finitely presented Gorenstein-projective $\mathcal{S}$-modules. That is, $\mathcal{S}$-modules (in general infinite dimensional) which have finite presentations $P_1 \to P_0 \to M \to 0$ with P_0, P_1 finite dimensional projective, and $\mathrm{Ext}^1_{\mathcal{S}}(M, P) = 0$ for every finite dimensional projective module P.

One proves that this is a Frobenius category and so has an associated stable category $\underline{\mathsf{gpr}}(\mathcal{S})$ which is canonically isomorphism to $\mathcal{D}_Q$. Moreover, $\mathsf{gpr}(\mathcal{S})$ is isomorphic to the category of finite dimensional projective $\mathcal{R}$-modules. There is a functor $\Omega : \mathsf{mod}\text{-}\Sigma \to \underline{\mathsf{gpr}}(\mathcal{S})$, and the composition gives the functor Φ.