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NOTETAKER CHECKLIST FORM

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Name: Stephen Hermes **Email/Phone:** SRHERMES @ BRANDEIS. EDU.

Speaker's Name: P.-G. Plamondon

Talk Title: Independence for Exchange Graphs & Cluster Complexes

Date: 10 / 29 / 12 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Cluster algebras, cluster categories, geometric type, exchange graph, cluster character

Please summarize the lecture in 5 or fewer sentances: The speaker introduced a conjecture of Fomin-Zelevinsky that the exchange graph of a cluster algebra is independent of coefficients. He introduced the notion of index to prove this conjecture for algebras of geometric type.

CHECK LIST

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Independence for Exchange Graphs and Cluster Complexes

P.-G. Plamondon October 29, 2012

Joint work with G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso.

1 Exchange Graphs and Cluster Complexes

Setting: Skew symmetric cluster algebras (i.e. the matrix B is symmetric) of geometric type. Can be encoded with an ice quiver Q without loops or 2-cycles. Assume the vertices are

$$\underbrace{1, 2, \ldots, r}_{\text{mutable}}, \underbrace{r+1, r+2, \ldots, n}_{\text{frozen}}.$$

Denote by Q_0 the full subquiver of Q containing the mutable vertices. This gives a cluster algebra $A(Q)$ of geometric type obtained by mutating at mutable vertices only. In this setting clusters look like

$$\{ \underbrace{u_1, \ldots, u_r}_{\text{cluster variables}}, \underbrace{x_{r+1}, \ldots, x_n}_{\text{coefficients}} \}.$$

Definition 1.1 (Fomin-Zelevinksy)**.** The *exchange graph* of $A(Q)$ is the graph whose vertices are clusters of $A(Q)$ (considered up to permutation) and two clusters are joined by an edge if and only if they differ by a mutation. (In general, the vertices are taken to be the seeds. It is a theorem that in this setting seeds are determined by their clusters.)

The *cluster complex* is a simplicial complex whose vertices are cluster variables (no coefficients) and whose simplices are subsets of clusters.

Example 1.2. For the quiver $1 \to 2$, the exchange graph is:

$$\begin{tikzcd}
& x_1 \arrow[r, dash] \arrow[dl, dash] & x_2 \arrow[dr, dash] & \\
\frac{1+x_1}{x_2} \arrow[drr, dash] & & & \frac{1+x_2}{x_1} \arrow[dl, dash] \\
& & \frac{1+x_1+x_2}{x_1 x_2} &
\end{tikzcd}$$

For the ice quiver

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & 2 \arrow[d] \\
{[3]} & {[4]}
\end{tikzcd}$$

(the [ ] denote the vertex is frozen) the exchange graph is:

$$\begin{tikzcd}
& x_1 \arrow[rr, dash] \arrow[dl, dash] & & x_2 \arrow[dr, dash] & \\
\dfrac{x_4\cdot 1+x_1}{x_2} \arrow[drr, dash] & & & & \dfrac{1+x_3\cdot x_2}{x_1} \arrow[dll, dash] \\
& & \dfrac{x_4\cdot 1+x_1+(x_3x_4)\cdot x_2}{x_1x_2} & &
\end{tikzcd}$$

Note the cluster complex doesn't change if we add frozen vertices, though the variables do change. In particular, if we specialize the frozen variables to 1, we get the original quiver.

Conjecture 1.3 (FZ)**.** The exchange graph and cluster complex of $A(Q)$ only depend on Q_0. (Originally conjectured in full generality, not just geometric type and skew-symmetric.)

Has been proven for skew-symmetrizable algebras by Gekhtman-Shapiro-Vainshtein provided the defining matrix B is of full rank.

Theorem 1.4 (CKLP)**.** The conjecture holds for any skew-symmetric cluster algebra.

Strategy: On one side we have the exchange graph and the cluster complex. From this we build an exchange graph and cluster complex of "indices". To prove the equivalence of these two we need categorification.

2 Additive Categorification via Cluster Categories

To define a cluster category we need not just a quiver Q but a potential W (in the sense of [DWZ]). From this C. Amiot defines a triangulated category $\mathcal{C}_{Q,W}$ called the cluster category generalizing the cluster category of [BMRRT]. It has the following properties:

1. It is a $\mathbb{C}$-additive triangulated category with suspension functor [1].
2. It comes with an object $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_n$.
3. Γ is rigid, i.e., $\mathrm{Hom}_{\mathcal{C}}(\Gamma, \Gamma[1]) = 0$.
4. (Keller-Yang) There is a notion of mutation in $\mathcal{C}_{Q,W}$: For each $1 \le i \le n$ can replace Γ_i with Γ_i^* to get a new object $\mu_i\Gamma$ and likewise for all the resulting objects.

Remark 2.1. This category does depend on the frozen vertices.

Definition 2.2. The objects $T = T_1 \oplus \cdots \oplus T_r \oplus T_{r+1} \oplus \cdots \oplus T_n$ obtained by mutating Γ are the *reachable cluster-tilting objects*. Their indecomposable summands are called *indecomposable reachable*.

Theorem 2.3 (Caldero-Chapoton (Dynkin Type); Caldero-Keller (acyclic); Palu (Hom-finite); P. (Hom-infinite); CKLP)**.** There is an (explicit) bijection

$$\{\text{indecomposable reachable object in } \mathcal{C}_{Q,W}\}/\text{iso} \to \{\text{cluster variables in } A(Q)\}$$

sending $\Gamma_i \mapsto x_i$ which commutes with mutation.

Definition 2.4. The *exchange graph* of $\mathcal{C}_{Q,W}$ is the graph whose vertices are the reachable cluster-tilting objects (up to isomorphism) and edges given by mutation.

The *reachable complex* of $\mathcal{C}_{Q,W}$ is the simplicial complex whose vertices are the reachable cluster-tilting objects, and so on.

Corollary 2.5. The exchange graph (resp. cluster complex) of $A(Q)$ is isomorphic to the exchange graph (resp. reachable complex) of $\mathcal{C}_{Q,W}$.

3 Indices

Properties of reachable objects:

1. Any reachable object is rigid.
2. Any reachable object X sits in a triangle

$$T_1^X \to T_0^X \to X \to T_1^X[1]$$

with T_1^X and T_0^X are direct sums of direct summands of Γ (i.e. lie in the additive closure $\mathsf{add}\ \Gamma$ of Γ.)

Definition 3.1. The *index* of X is

$$\mathrm{ind}_\Gamma X = [T_0^X] - [T_1^X] \in K_0(\mathsf{add}\ \Gamma) = \mathbb{Z}[\Gamma_1] \oplus \cdots \oplus \mathbb{Z}[\Gamma_n]$$

It is equal to $\sum_{i=1}^n g_i^X [\Gamma_i]$ for some g_i^X.

Proposition 3.2 (Dehy-Keller, P.)**.** Any reachable (in fact, rigid) object is determined by its index.

Definition 3.3. The *exchange graph* of indices is the graph obtained from the exchange graph of $\mathcal{C}_{Q,W}$ by replacing each indecomposable reachable object by its index. The *complex of indices* is defined similarly.

Example 3.4. $0 \to \Gamma_i \to \Gamma_i \to 0[1] = 0$ so $\mathrm{ind}_\Gamma \Gamma_i = 1 \cdot [\Gamma_i] = (0, 0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0)$.

Example 3.5. For Q the quivers in Example **??**, the exchange graphs of indices are

$$\begin{tikzcd}
& (1,0) \arrow[rr, dash] \arrow[dl, dash] & & (0,1) \arrow[dr, dash] & \\
(0,-1) \arrow[drr, dash] & & & & (-1,1) \arrow[dll, dash] \\
& & (-1,0) & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (1,0,0,0) \arrow[rr, dash] \arrow[dl, dash] & & (0,1,0,0) \arrow[dr, dash] & \\
(0,-1,0,1) \arrow[drr, dash] & & & & (-1,1,1,0) \arrow[dll, dash] \\
& & (-1,0,1,1) & &
\end{tikzcd}$$

respectively. Note that the top of the indices are unchanged by adding frozen vertices.

Thus proving that adding frozen vertices has the effect of appending entries to the original index exchange graph would prove the conjecture.

Proposition 3.6 (CKLP)**.** The indecomposable reachable objects depend only on and are determined by the mutable part of their index .

Corollary 3.7. The conjecture holds for any $A(Q)$.