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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** Stephen Hermes **Email/Phone:** SKHERMES@BRANDEIS.EDU

**Speaker's Name:** Hugh Thomas

**Talk Title:** (Higher-Dimensional Tropical) Cluster Algebras in Combinatorics, Algebra, and (Convex) Geometry

**Date:** 10 / 29 / 12 **Time:** 3 : 00 am / [✓ pm] (circle one)

**List 6-12 key words for the talk:** Cluster algebras, surface cluster algebras, g-vectors, cyclic polytopes, 2d-CY algebras, Auslander Algebras, tropical geometry

**Please summarize the lecture in 5 or fewer sentances:** The speaker introduced a family of algebras generalizing the cluster algebras of Type An. He introduced tropical exchange relations for these algebras and showed analogues of g-vectors obeyed these relations. He then introduced a categorification of these algebras.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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# (Higher-Dimensional Tropical) Cluster Algebras in Combinatorics, Algebra, and (Convex) Geometry

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Hugh Thomas October 29,2012

Joint with Steffen Oppermann; arXiv:1001.5437 and subsequent work.

## 1 A Clusterlike Algebra

Associated to a quiver of type $A_n$ we have two 2-d objects: a 2-CY cluster category associated, and the 2-dimensional disk (with marked points on the boundary). We want to replace the category with a generalized cluster category associated to a higher Auslander algebra of linear $A_n$ and the disk with a $2d$-dimensional cyclic polytope.

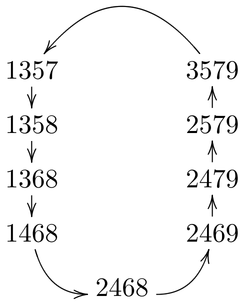
Pick $d$ odd (for simplicity) and an integer $n \geq 2d+2$ ($d = 1$ is classical). Let $I$ be the set of $(d+1)$-sets from $\{1, \ldots, n\}$. The *cluster variables* are the formal variables $x_A$ for $A \in I$. We say $A = \{a_0, \ldots, a_d\}$ and $B = \{b_0, \ldots, b_d\}$ *intertwine* (denoted $A \wr B$) iff $a_0 < b_0 < a_1 < b_1 < \cdots < a_d$ and either $b_d > a_d$ or $b_d < a_0$.

Two cluster variables $x_A$ and $x_B$ are *incompatible* iff $A \wr B$. The *clusters* are the maximal collections of pairwise compatible cluster variables. This maximal size is $\binom{n-d-1}{d} + \binom{n}{d+1} - \binom{n-d}{d+1}$. Let $\mathcal{C}$ be the set of clusters.

If $\{i, i+1\} \subset A$ for $A \in I$, then $x_A$ is compatible with everything. Let $I^f$ (for frozen) be the set of such $A$ and $I^m = I - I^f$ (for mutable).

If $n = 2d+2$, then $I^m = \{\{1, 3, \ldots\}, \{2, 4, \ldots, \}\}$. If $n = 2d+3$, then $|I^m| = n$.

**Example 1.1.**



Any four consecutive subsets form a cluster, and mutation is given by swapping out the subset at the end for the net one at the other end. Not every set of three subsets can be completed to a cluster; e.g. 1357, 1468, 2479 can't be completed to a cluster.

If $A$ and $B$ intertwine, then $x_A$ and $x_B$ are exchangable. That is, there exists a set of indices in $I$ that if you add in $A$ or add in $B$ you get a cluster.

**Theorem 1.2** (Rambau, after some reformulation)**.** The exchange graph (defined in the obvious way) is connected.

### 1.1 Tropical Exchange Relations

Given sets $A, B \in I$ with $A \wr B$, and a subset $J \subset \{0, \dots, d\}$, define $n(J, A, B) = \{a_i : i \notin J\} \cup \{b_i : i \in J\}$ and $m(J, A, B) = \{a_i : i \notin J\} \cup \{b_{i-1} : i \in J\}$ where $b_{-1} := b_d$.

The *exchange relation* is

$$x_A + x_B = \max \left\{ \sum_{J \text{ proper subset}} (-1)^{|J|+1} x_{n(J,A,B)}, \sum_{J \text{ proper subset}} (-1)^{|J|+1} x_{m(J,A,B)} \right\}.$$

**Theorem 1.3.** Let $\mathcal{K}$ be the set of $(d+1)$-subsets of $\mathbb{R} - \{1, \dots, n\}$ and $\mathcal{L} = \{L : L \text{ is a compatible subset of } \mathcal{K}\}$ (called *laminations*). given $L \in \mathcal{L}$ and $A \in I$ define $P_A^L = |\{B \in L : A \wr B\}|$. Then $x_A := P_A^L$ satisfies the exchange relation.

### 1.2 $g$-Vectors

Classically, one writes $g$-vectors in the following way. Start with a $B$-matrix $B^0$ and a starting cluster $t_0$. For a target cluster $t$ and target cluster variable $l$ we get a number $g_{l,t}$.

For $T \in \mathcal{C}$ and $D \in I^m$, define

$$g_D^T = \sum_{B \in T^m} c_B y_B$$

where $c_B$ is a number and $y_B$ is a new set of formal variables. If $D \in T$, then set $g_D^T = y_D$. Suppose $S, T$ are clusters and $S - \{A\} = T - \{B\}$. Then

$$g_D^S = \begin{cases} g_D^T|_{y_B = g_B^S} & \text{if} c_B > 0 \\ g_D^T|_{y_B = -g_{\Sigma B}^S} & \text{if} c_B < 0 \end{cases}.$$

where if $B = \{b_0, \dots, b_d\}$, we define $\Sigma B = \{b_0 + 1, \dots, b_d + 1\}$.

## 2 Categorification

The output is a category $\mathcal{O}$ with $\mathsf{ind}\ \mathcal{O} = \{M_A : A \in I^m\}$. It is $2d$-CY with shift $\Sigma = [d]$. It is $(d+2)$-angulated.

An object $T \in \mathcal{O}$ is cluster tilting if $\operatorname{Ext}^d(T, T) = 0$ (all lower Ext's vanish automatically). Any object $X$ of $\mathcal{O}$ occurs in a $(d+2)$-angle

$$X[-d] \to T_d \to \cdots \to T_0 \to X \to$$

where each $T_i \in \mathsf{add}\ T$.

**Theorem 2.1** (OT)**.** Basic cluster tilting objects in $\mathcal{O}$ correspond to clusters in $\mathcal{C}$ bijectively. We can define the *index* of $X$ by

$$i_T(X) = [T_0] - [T_1] + [T_2] - \cdots \pm [T_d]$$

where $[T] = \sum_{A \in I^m} [T : M_A] y_A$ for some numbers $[T : M_A]$. With this index, $i_T(M_A) = g_A^T$.