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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SPHERMES @ BRANDEIS. EDU

Speaker's Name: R. Kenyon

Talk Title: The Garnet Recurrence & The Ising Model

Date: 10 / 30 / 12 **Time:** 11 : 20 am / pm (circle one)

List 6-12 key words for the talk: Statistical Mechanics, Ising model, Octahedral recurrence, cluster mutation, Laurent phenomenon

Please summarize the lecture in 5 or fewer sentances: The speaker discussed several recurrences exhibiting cluster-mutation-like properties. He shows these recurrences satisfy the Laurent phenomenon and gave combinatorial classifications of the Laurent monomials. These combinatorial classifications show a large scale regularity.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

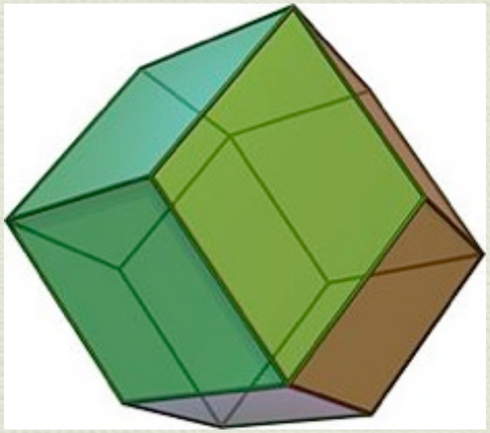
- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

The Garnet Recurrence and the Ising Model

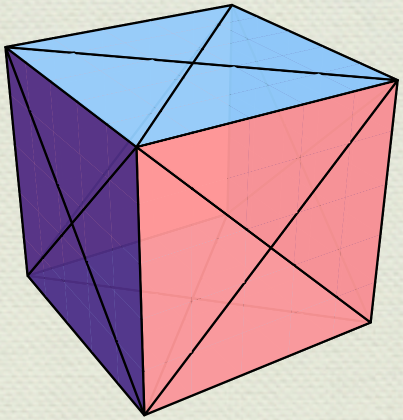
R. Kenyon (Brown)

R. Pemantle (UPenn)

garnet crystal



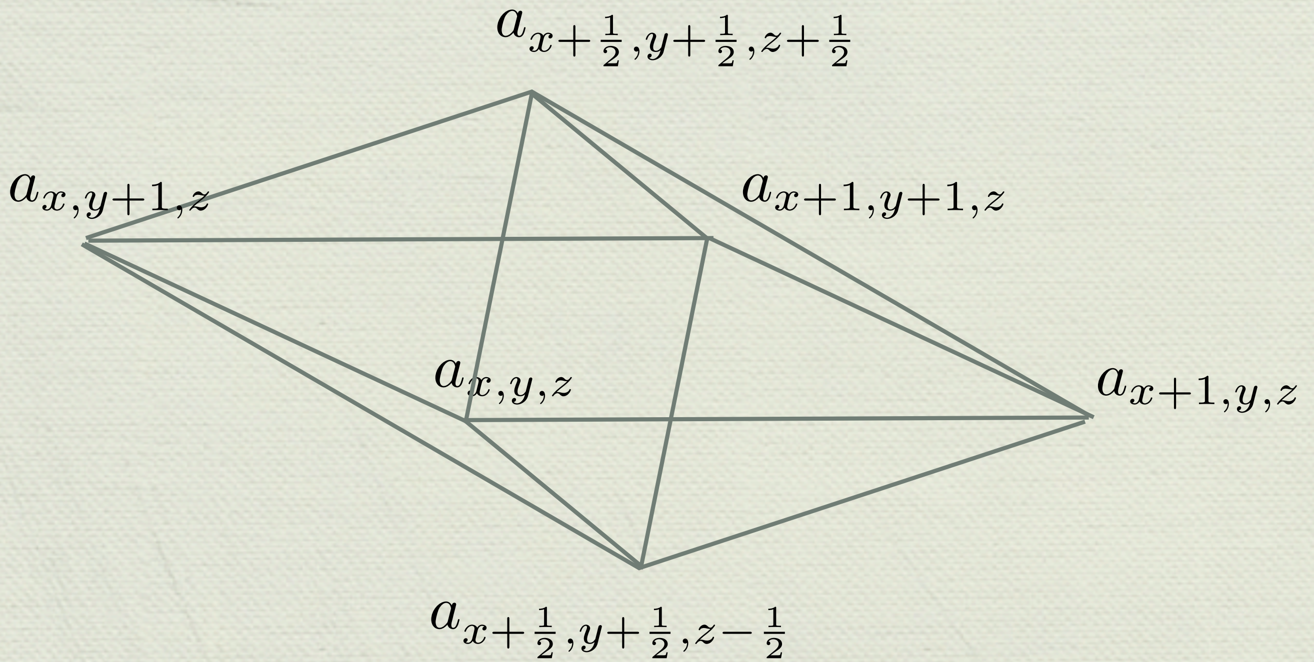
rhombic dodecahedron

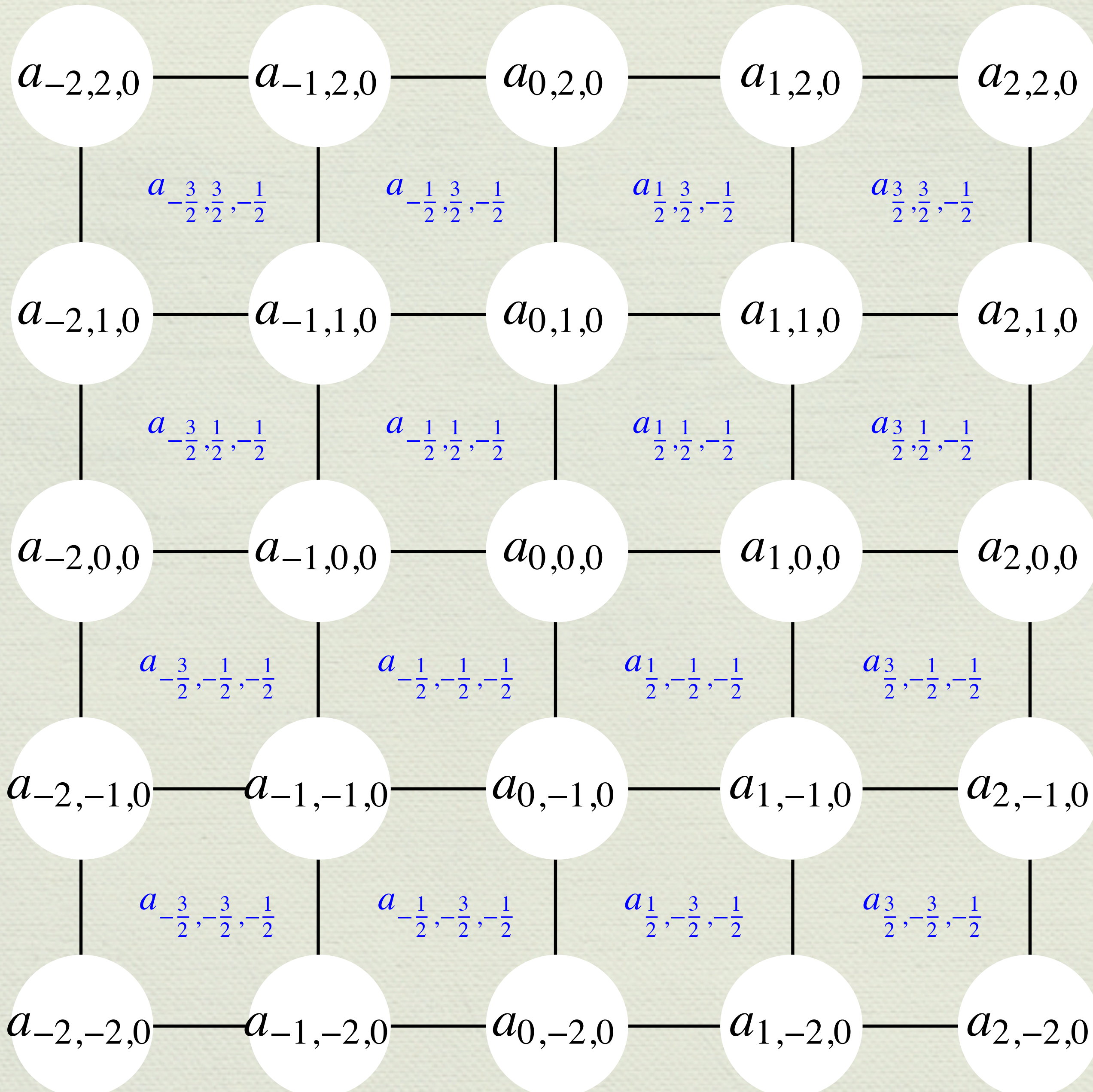


Octahedron recurrence = Hirota bilinear difference equation (HBDE)

$$a_{x+\frac{1}{2},y+\frac{1}{2},z+\frac{1}{2}} = \frac{a_{x,y,z}a_{x+1,y+1,z} + a_{x,y+1,z}a_{x+1,y,z}}{a_{x+\frac{1}{2},y+\frac{1}{2},z-\frac{1}{2}}}$$

diamond crystal





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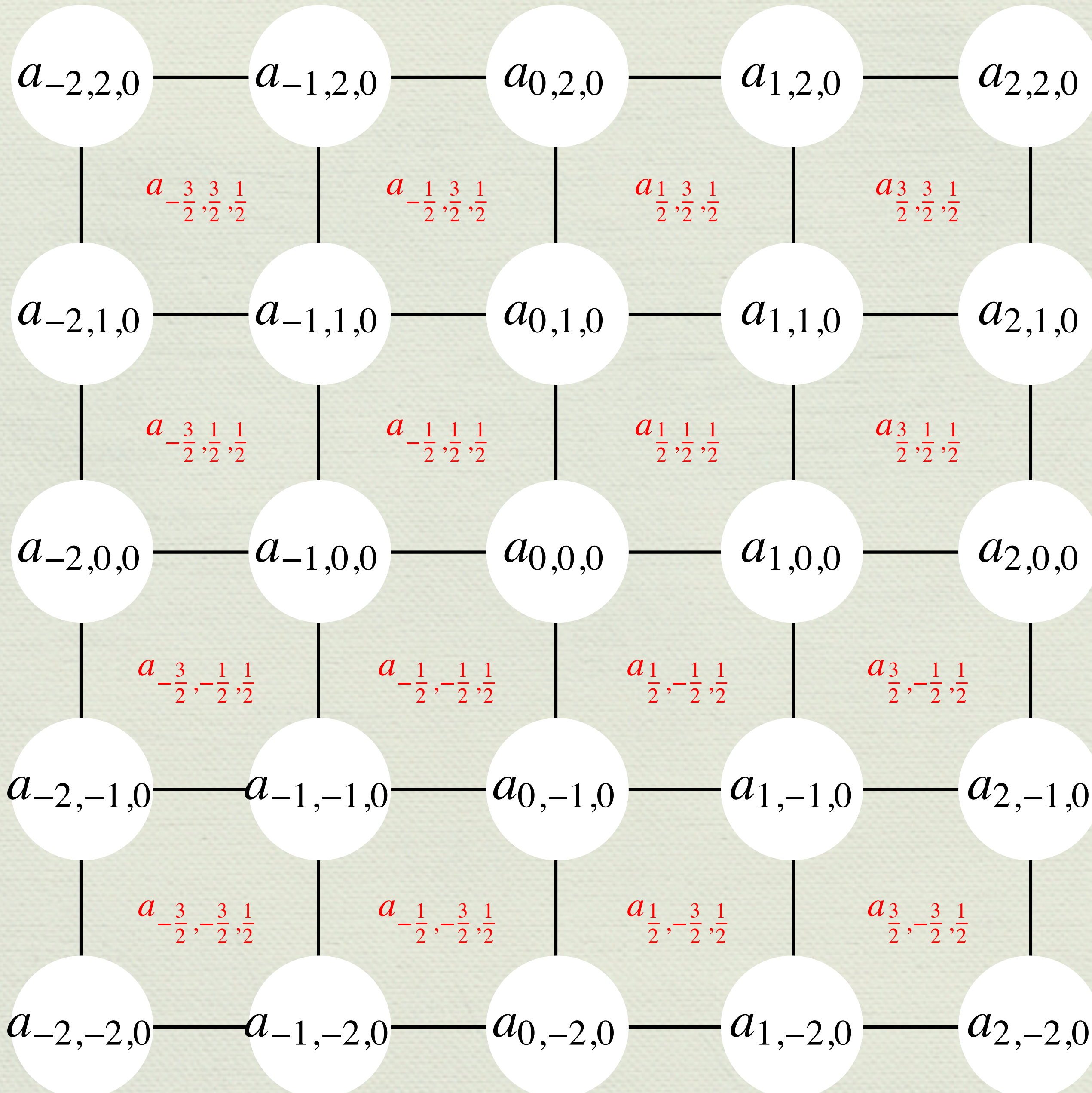
$a_{-2,2,0}$ $a_{-1,2,0}$ $a_{0,2,0}$ $a_{1,2,0}$ $a_{2,2,0}$
$a_{-\frac{3}{2},\frac{3}{2},\frac{1}{2}}$ $a_{-\frac{1}{2},\frac{3}{2},\frac{1}{2}}$ $a_{\frac{1}{2},\frac{3}{2},\frac{1}{2}}$ $a_{\frac{3}{2},\frac{3}{2},\frac{1}{2}}$
$a_{-2,1,0}$ $a_{-1,1,0}$ $a_{0,1,0}$ $a_{1,1,0}$ $a_{2,1,0}$
$a_{-\frac{3}{2},\frac{1}{2},\frac{1}{2}}$ $a_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}$ $a_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$ $a_{\frac{3}{2},\frac{1}{2},\frac{1}{2}}$
$a_{-2,0,0}$ $a_{-1,0,0}$ $a_{0,0,0}$ $a_{1,0,0}$ $a_{2,0,0}$
$a_{-\frac{3}{2},-\frac{1}{2},\frac{1}{2}}$ $a_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}}$ $a_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}$ $a_{\frac{3}{2},-\frac{1}{2},\frac{1}{2}}$
$a_{-2,-1,0}$ $a_{-1,-1,0}$ $a_{0,-1,0}$ $a_{1,-1,0}$ $a_{2,-1,0}$
$a_{-\frac{3}{2},-\frac{3}{2},\frac{1}{2}}$ $a_{-\frac{1}{2},-\frac{3}{2},\frac{1}{2}}$ $a_{\frac{1}{2},-\frac{3}{2},\frac{1}{2}}$ $a_{\frac{3}{2},-\frac{3}{2},\frac{1}{2}}$
$a_{-2,-2,0}$ $a_{-1,-2,0}$ $a_{0,-2,0}$ $a_{1,-2,0}$ $a_{2,-2,0}$

Dodgson condensation for computing determinants

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{array}{ccccc} a & & b & & c \\ & ae-bd & & bf-ce & \\ d & & e & & f \\ & dh-eg & & ei-fh & \\ g & & h & & i \end{array}$$

Dodgson condensation for computing determinants

$$\det \begin{pmatrix} a & & b & & c \\ & 1 & & 1 & \\ d & & e & & f \\ & 1 & & 1 & \\ g & & h & & i \end{pmatrix}$$

$$\begin{array}{ccccc} a & & b & & c \\ & ae - bd & & bf - ce & \\ d & & e & & f \\ & dh - eg & & ei - fh & \\ g & & h & & i \end{array}$$

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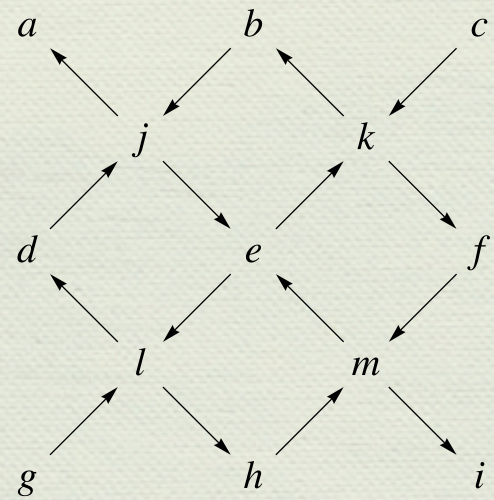
Dodgson condensation for computing determinants

$$\det \begin{pmatrix} a & & b & & c \\ & 1 & & 1 & \\ d & & e & & f \\ & 1 & & 1 & \\ g & & h & & i \end{pmatrix}$$

$$\begin{array}{ccccc} a & & b & & c \\ & ae-bd & & bf-ce & \\ d & & e & & f \\ & dh-eg & & ei-fh & \\ g & & h & & i \end{array}$$

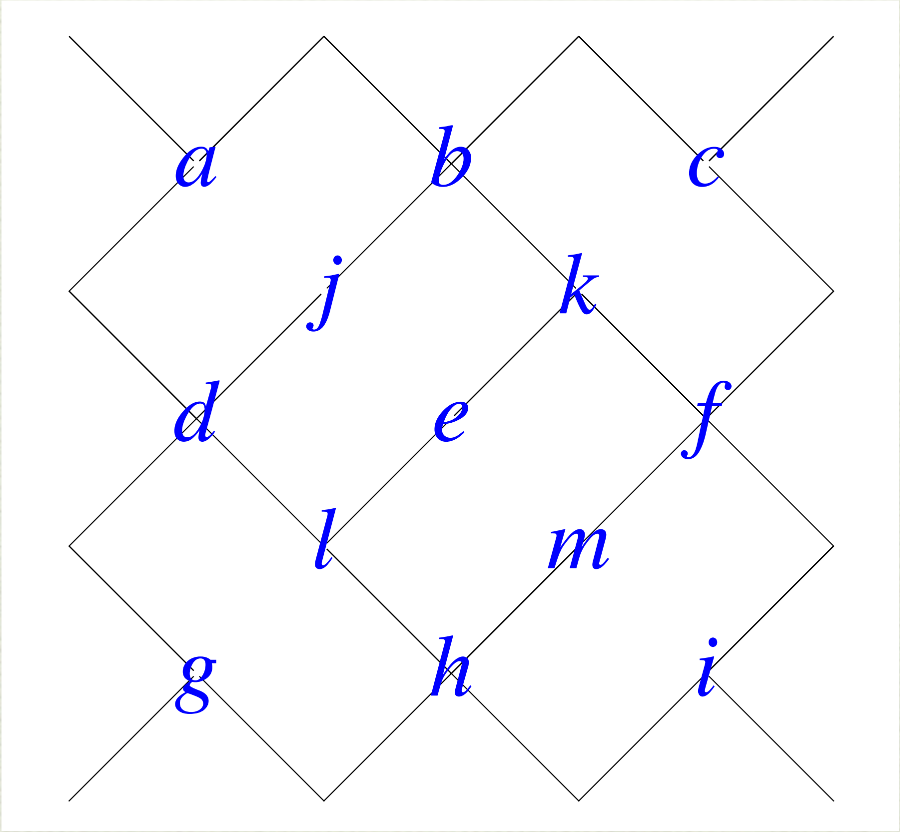
$$\frac{(ae-bd)(ei-fh)-(bf-ce)(dh-eg)}{e} = \det M$$



The octahedron recurrence is a "cluster modular transformation"

$$e_1 = \frac{aei}{jm} + \frac{bdi}{jm} + \frac{afh}{jm} + \frac{bdfh}{ejm} + \frac{bdfh}{ekl} + \frac{bfg}{kl} + \frac{cdh}{kl} + \frac{ceg}{kl}$$

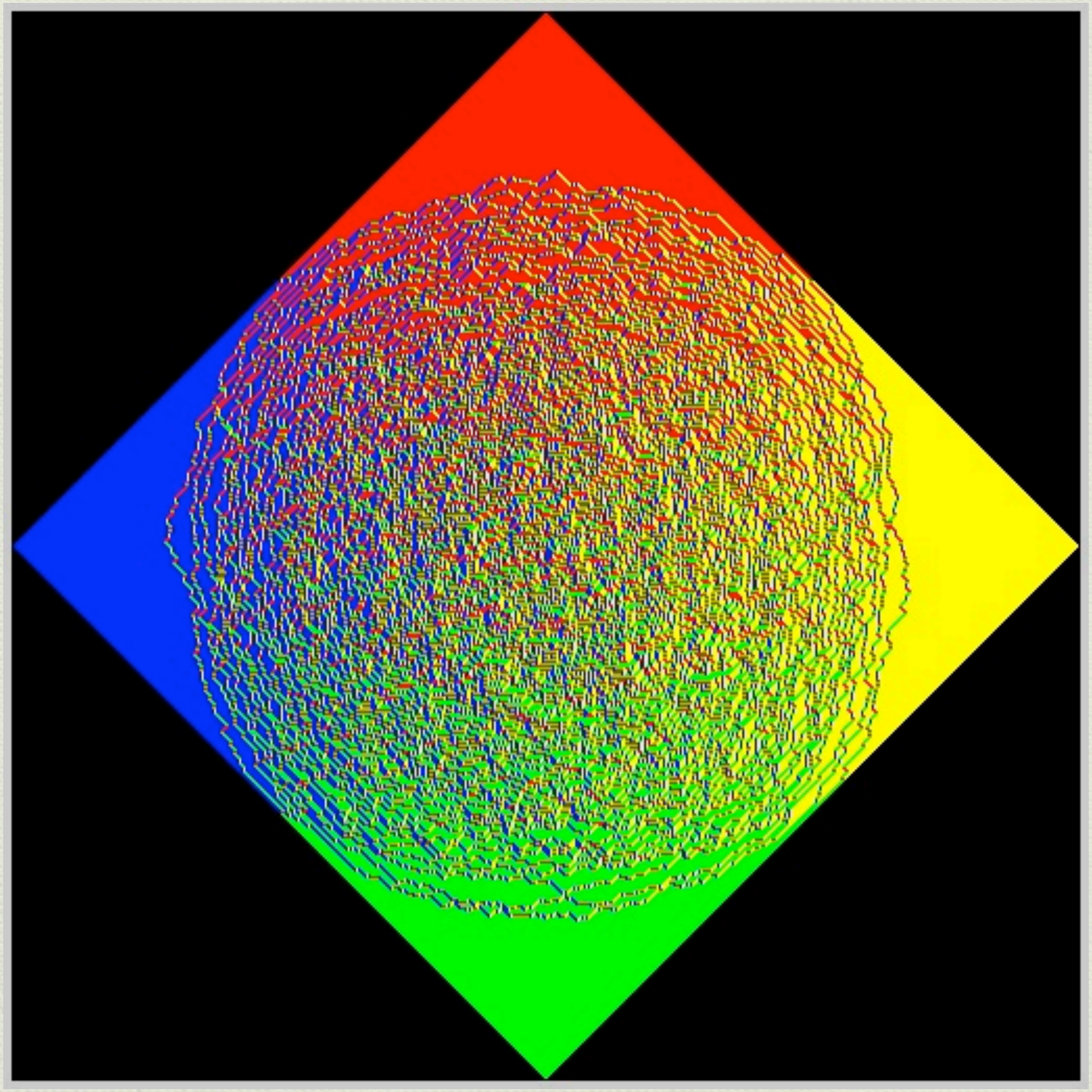
Theorem [Fomin,Zelevinsky]:
$a_{i,j,k}$ is a Laurent polynomial in the initial variables ($k = 0, -1/2$).



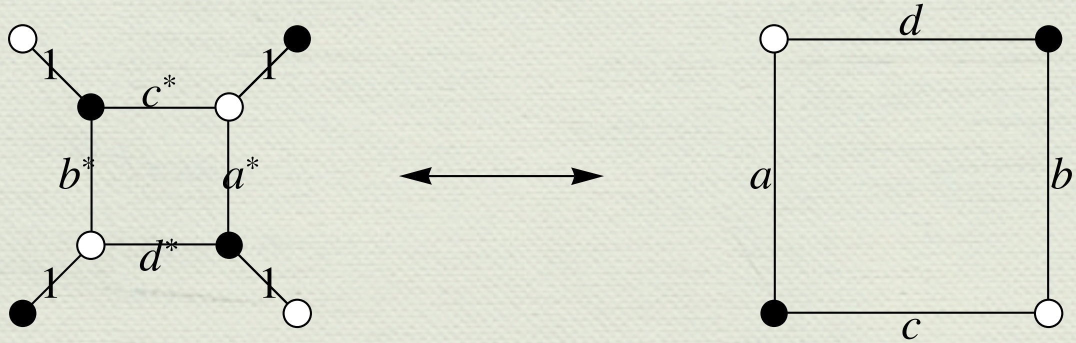
Tilings of the Aztec diamond.

(Speyer)

$$a_{0,0,1} = \frac{aei}{jm} + \frac{bdi}{jm} + \frac{afh}{jm} + \underbrace{\frac{bdfh}{ejm}} + \frac{bdfh}{ekl} + \frac{bfg}{kl} + \frac{cdh}{kl} + \frac{ceg}{kl}$$



The proof is based on an operation on (weighted) graphs called "urban renewal".



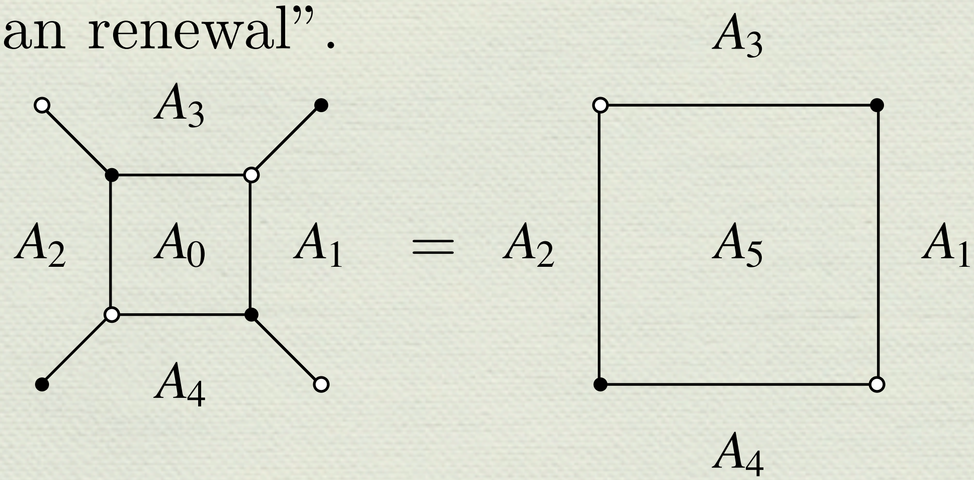
$$a^* = \frac{a}{ab+cd}$$

$$b^* = \frac{b}{ab+cd}$$

$$c^* = \frac{c}{ab+cd}$$

$$d^* = \frac{d}{ab+cd}$$

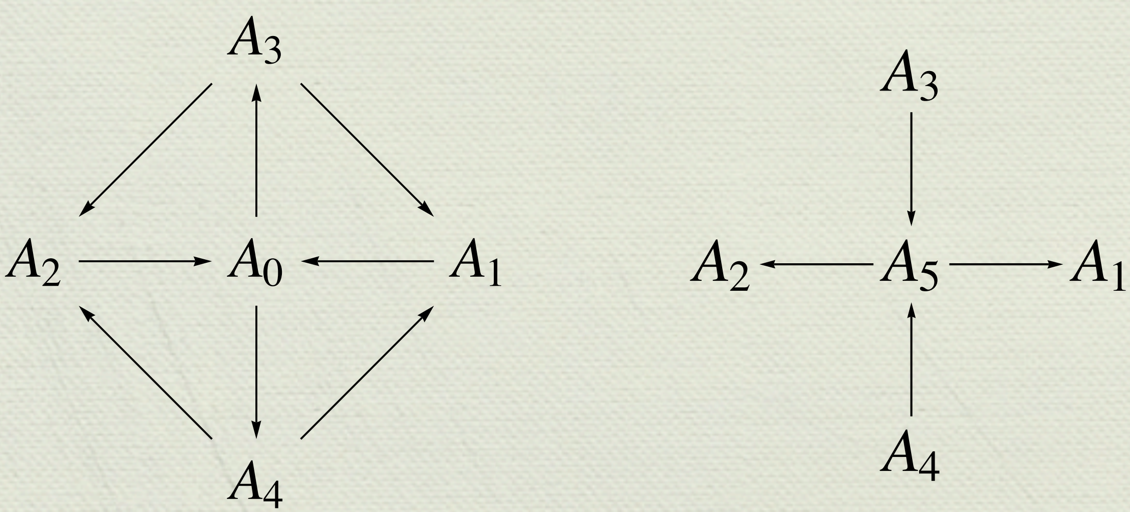
The proof is based on an operation on (weighted) graphs called "urban renewal".



(The A variables are related to the edge variables by: [*omitted*])

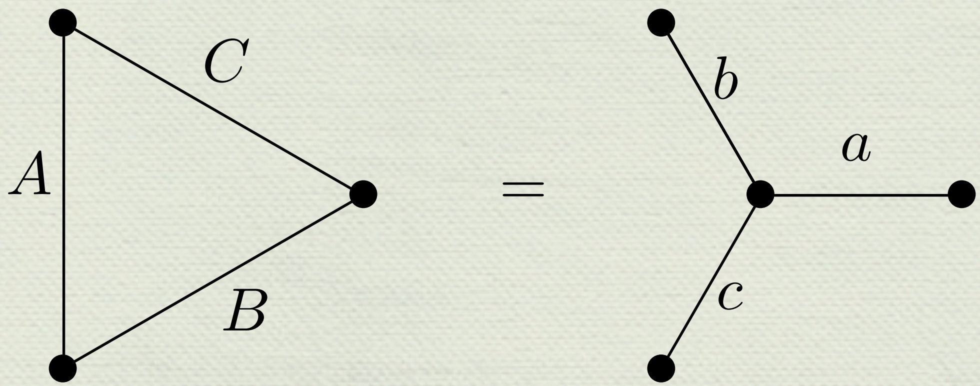
(Plücker coordinate vars)

In terms of quivers: mutation



Resistor networks

Y-Delta transformation for resistor networks

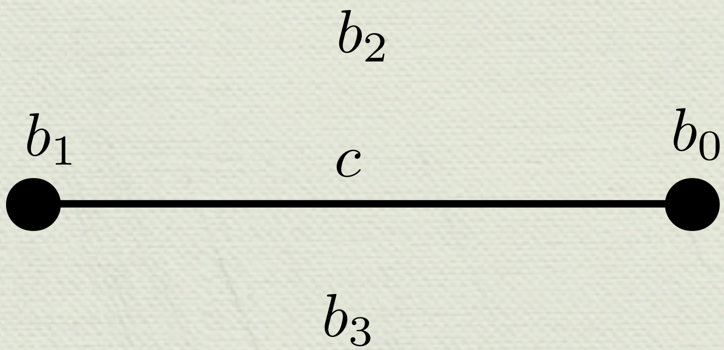


$$A = \frac{bc}{a+b+c}$$

$$B = \frac{ac}{a+b+c}$$

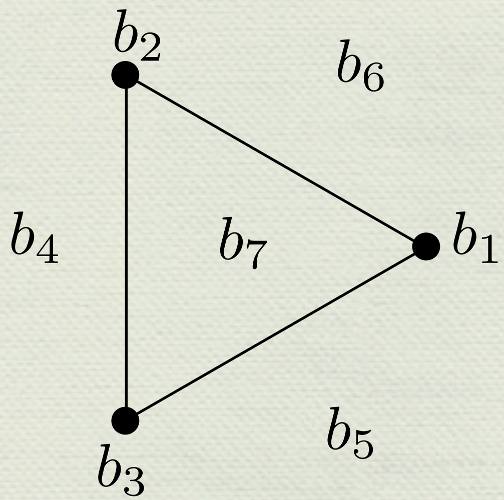
$$C = \frac{ab}{a+b+c}$$

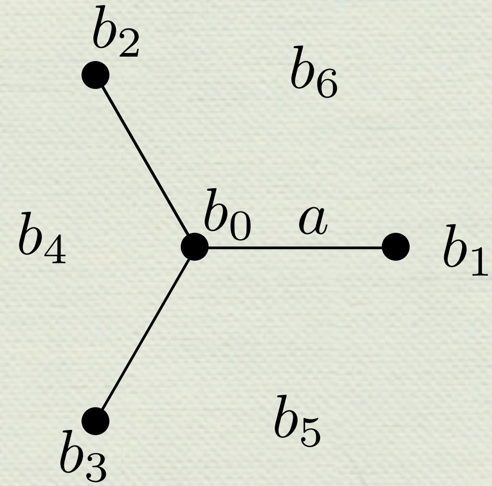
Define new coordinates ("activities") b_i on vertices and faces:



related to conductances by $c = \frac{b_0 b_1}{b_2 b_3}$.

Y-Delta transformation for resistor networks





where $a = \frac{b_0 b_1}{b_5 b_6}$ etc.

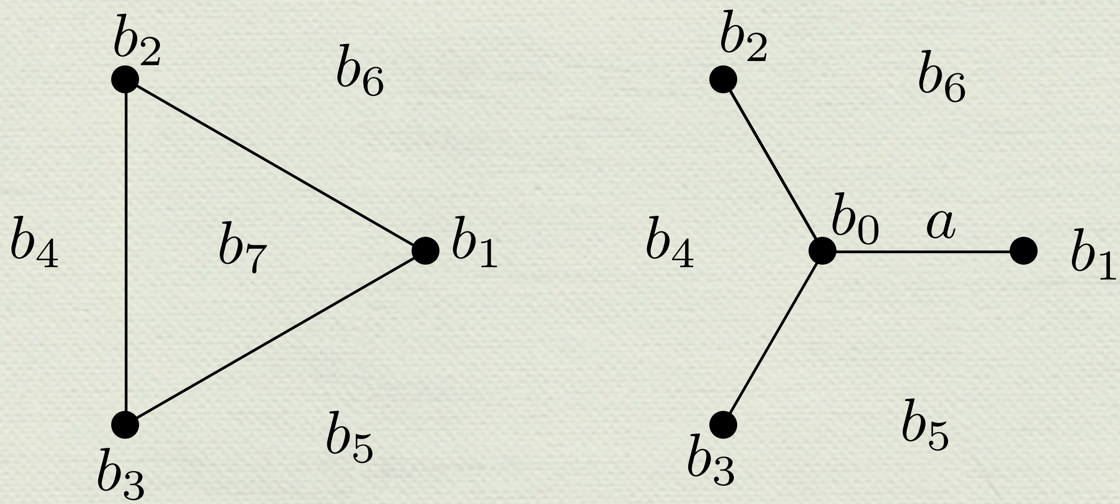
$$A = \frac{bc}{a+b+c}$$

Then $$b_0 b_7 = b_1 b_4 + b_2 b_5 + b_3 b_6.$$

This is the "cube recurrence":

$$b_{i+1,j+1,k+1} b_{i,j,k} = b_{i+1,j,k} b_{i,j+1,k+1} + b_{i,j+1,k} b_{i+1,j,k+1} + b_{i,j,k+1} b_{i+1,j+1,k}$$

Y-Delta transformation for resistor networks



where $a = \frac{b_0 b_1}{b_5 b_6}$ etc.

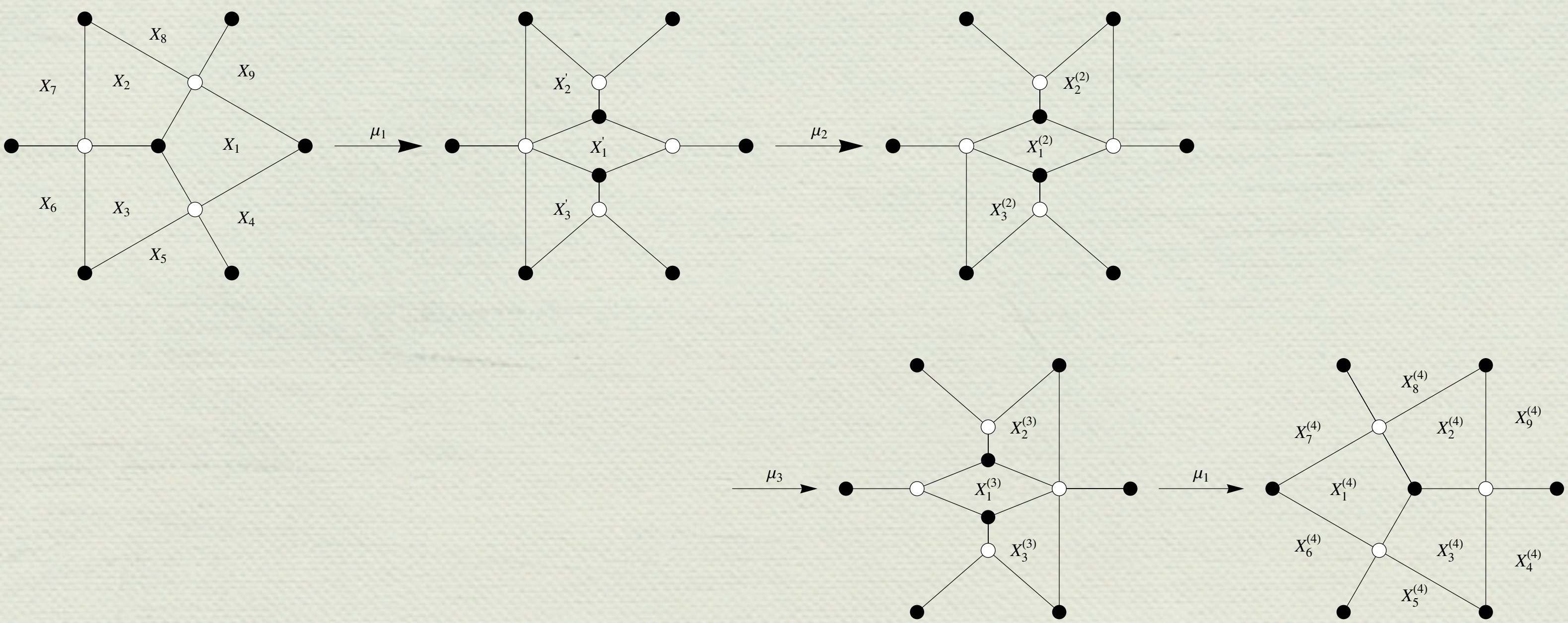
$$\frac{b_2 b_3}{b_4 b_7} = \frac{\frac{b_2 b_0}{b_4 b_6}\frac{b_0 b_3}{b_4 b_5}}{\frac{b_2 b_0}{b_4 b_6} + \frac{b_0 b_3}{b_4 b_5} + \frac{b_0 b_1}{b_5 b_6}}$$

Then $$b_0 b_7 = b_1 b_4 + b_2 b_5 + b_3 b_6.$$

This is the "cube recurrence":

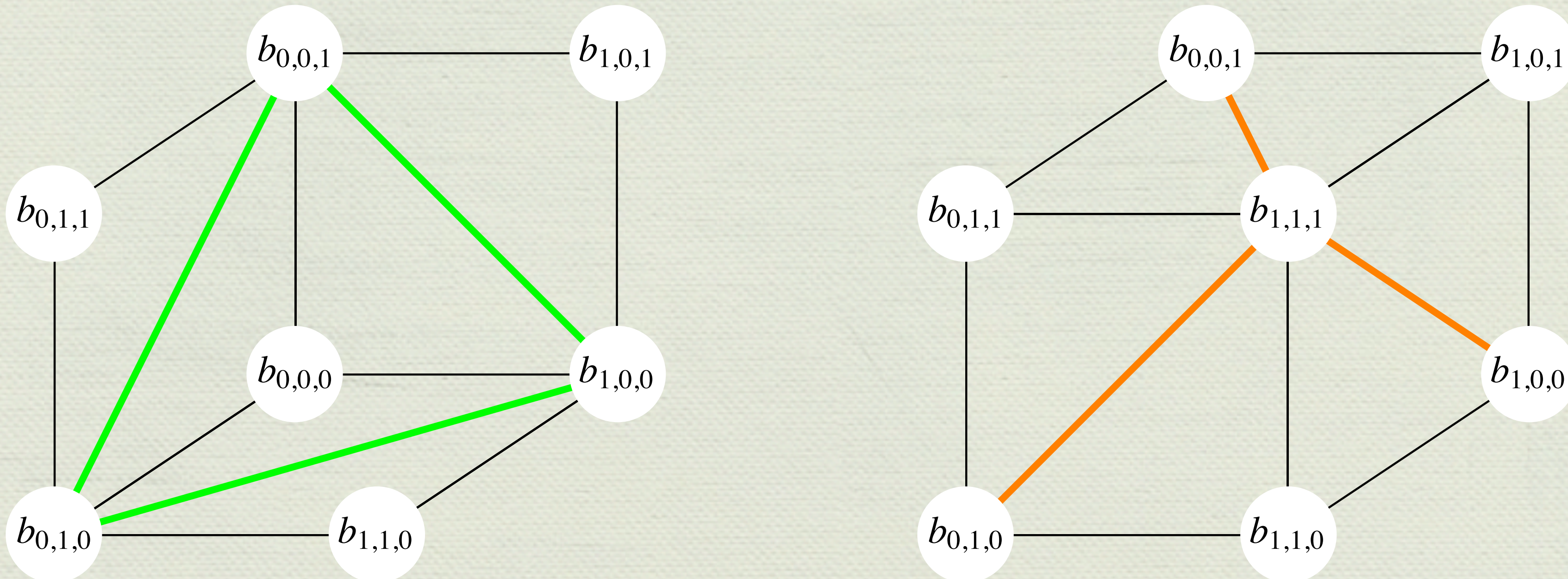
$$b_{i+1,j+1,k+1} b_{i,j,k} = b_{i+1,j,k} b_{i,j+1,k+1} + b_{i,j+1,k} b_{i+1,j,k+1} + b_{i,j,k+1} b_{i+1,j+1,k}$$

The cube recurrence



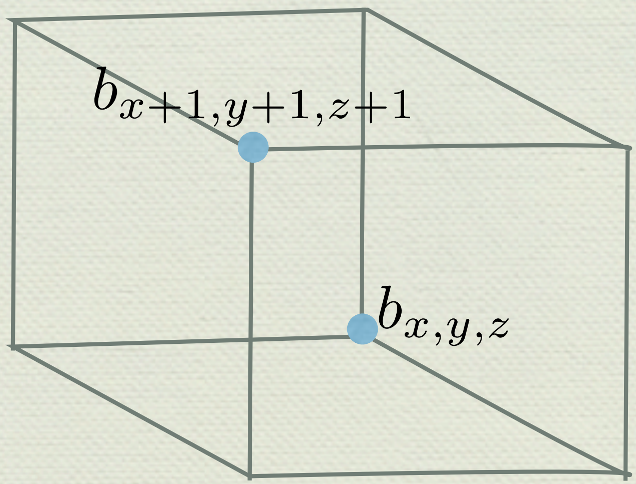
...is a composition of four urban renewals.

(four quiver mutations.)


$b_{0,0,1}$
$b_{1,0,1}$
$b_{0,1,1}$
$b_{0,0,0}$
$b_{1,0,0}$
$b_{0,1,0}$
$b_{1,1,0}$
$b_{0,0,1}$
$b_{1,0,1}$
$b_{0,1,1}$
$b_{1,1,1}$
$b_{1,0,0}$
$b_{0,1,0}$
$b_{1,1,0}$

Cube recurrence = Miwa equation

$$b_{x+1,y+1,z+1} = \frac{b_{x+1,y,z}b_{x,y+1,z+1} + b_{x,y+1,z}b_{x+1,y,z+1} + b_{x,y,z+1}b_{x+1,y+1,z}}{b_{x,y,z}}$$

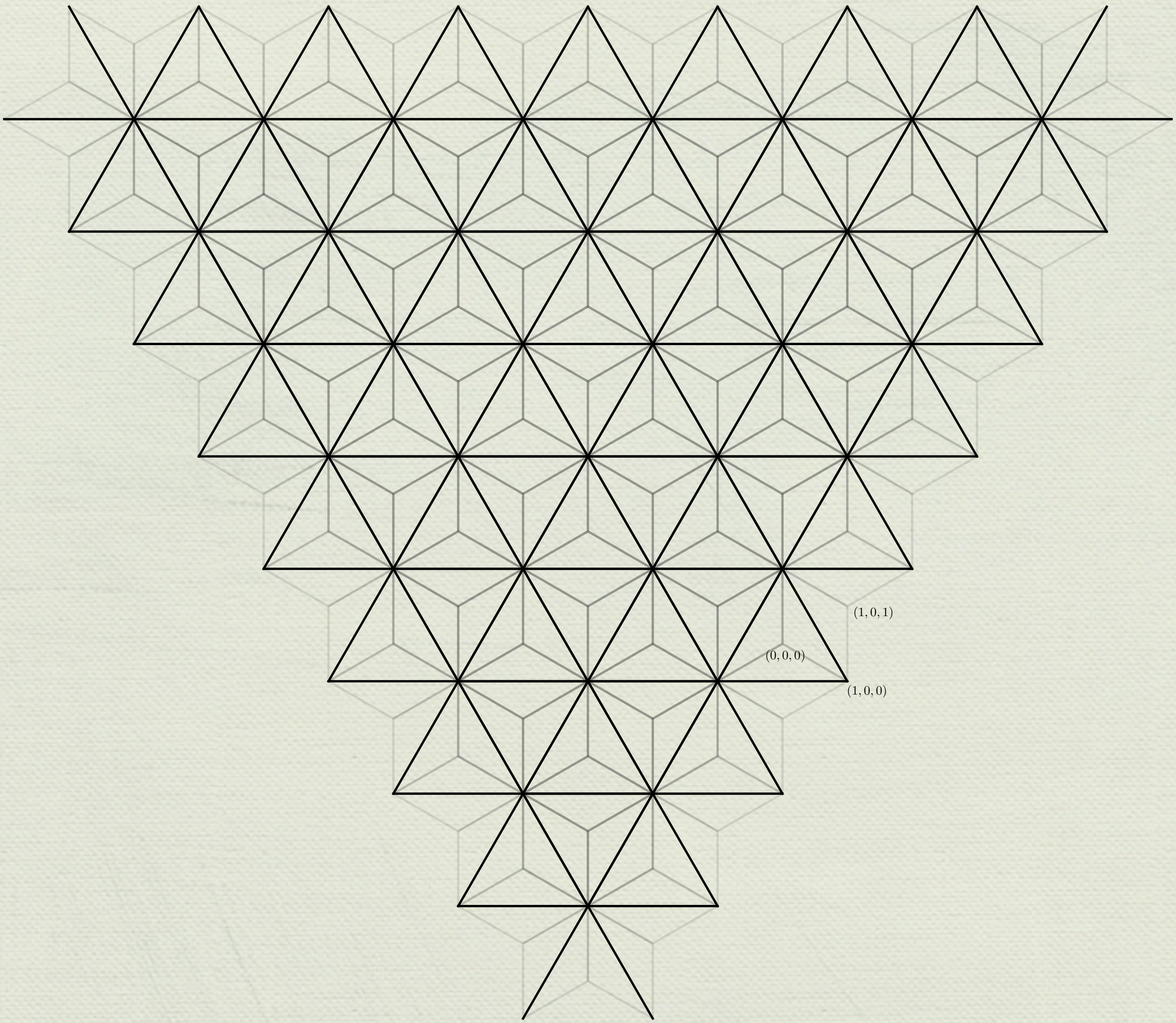


pyrite crystals

From the values on $0 \leq x + y + z \leq 2$ (or any other "stepped surface") we get all $b_{x,y,z}$.

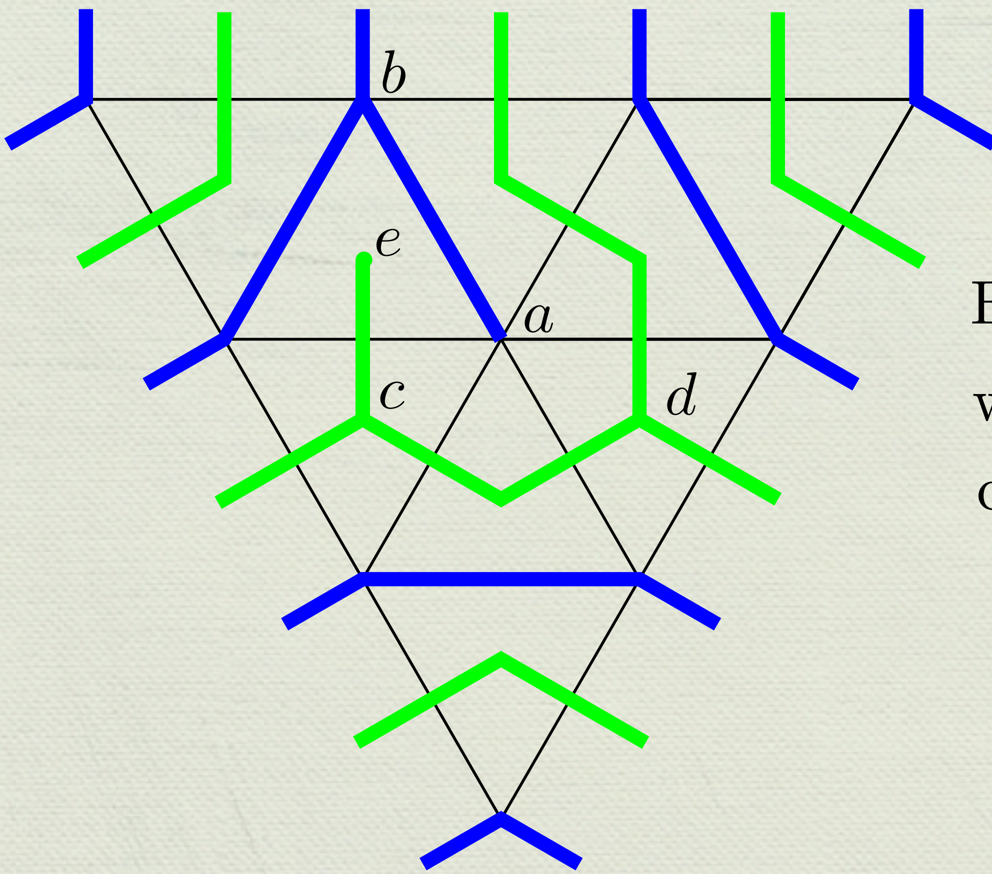
The Laurent property holds:

$b_{x,y,z}$ is a Laurent polynomial in the initial variables.


(1, 0, 1)
(0, 0, 0)
(1, 0, 0)

Terms in the Laurent expansion were identified by Carroll/Speyer.

$$b_{2,1,1} = \cdots + \frac{bd}{ace} + \ldots$$

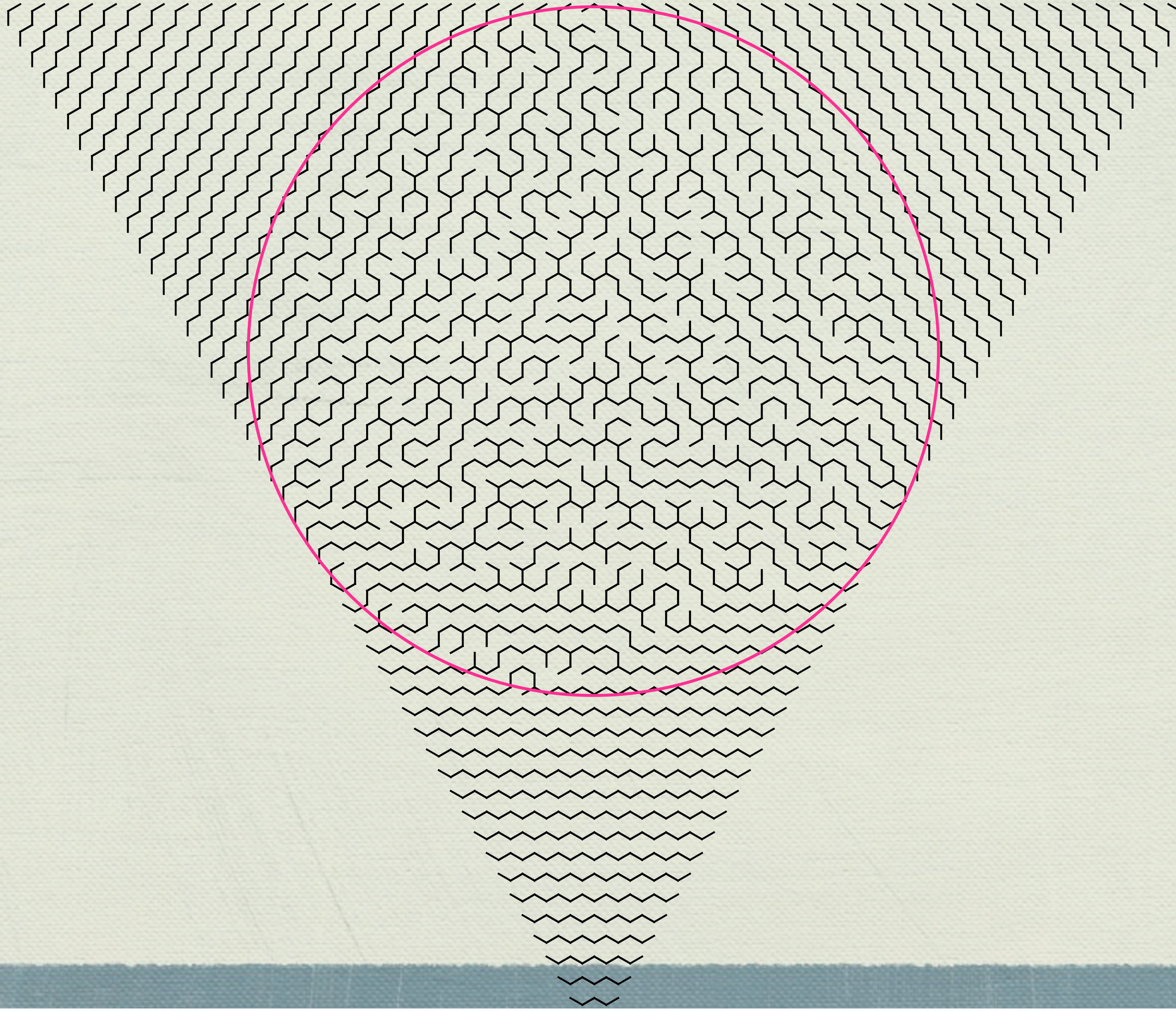


"Cube groves":

Essential spanning forests with "tripartite" boundary connections

Tuesday, October 30, 12

For large groves, arctic circle theorem [Peterson-Speyer]



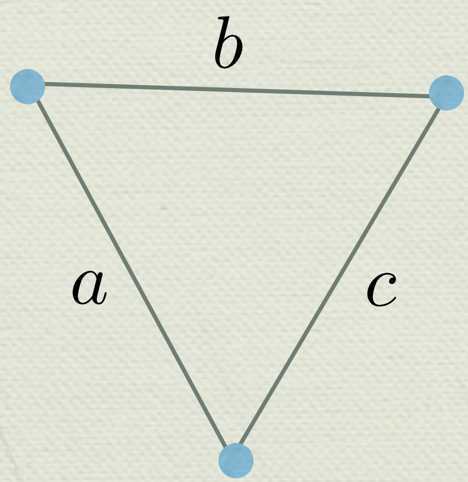
Ising model

G a graph, $c : E \to \mathbb{R}_{>0}$ edge weights.

Configuration space $\Omega = \{1, -1\}^G$

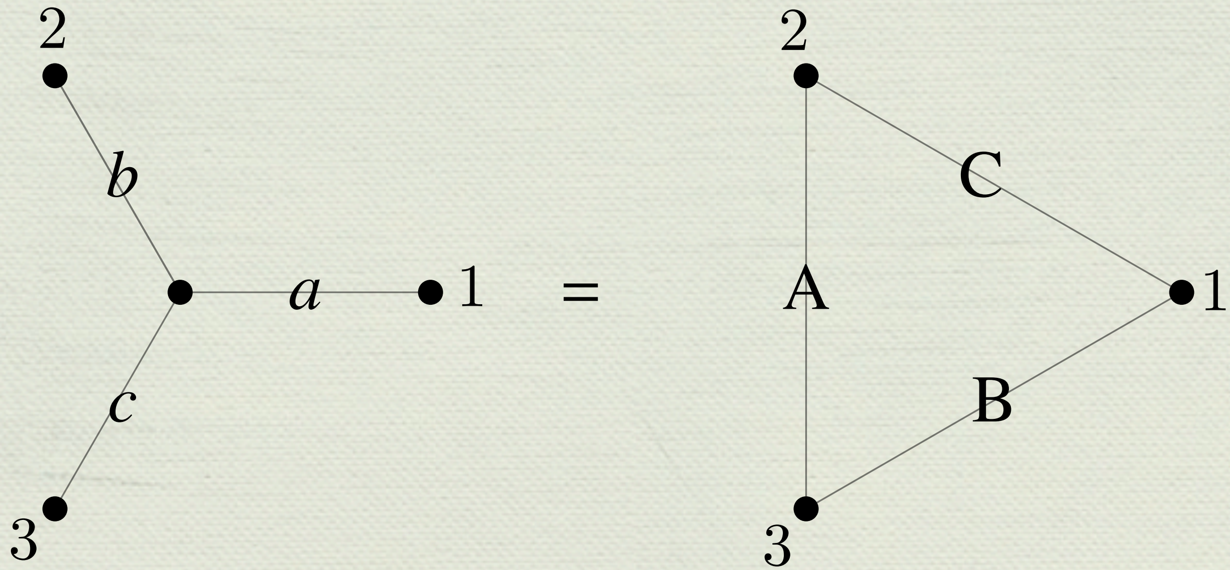
Partition function

$$Z = \sum_{\sigma \in \Omega} \prod_{i \sim j:\ \sigma_i = \sigma_j} c_{ij}$$



$$Z = 2abc + 2a + 2b + 2c$$

Ising model Y-Delta transformation



spins

1 2 3		
$+ + +$	$abc + 1$	ABC
$- + +$	$a + bc$	A
$+ - +$	$b + ac$	B
$+ + -$	$c + ab$	C

these should be proportional

"Before" and "after" are proportional (Ising measure preserved) iff

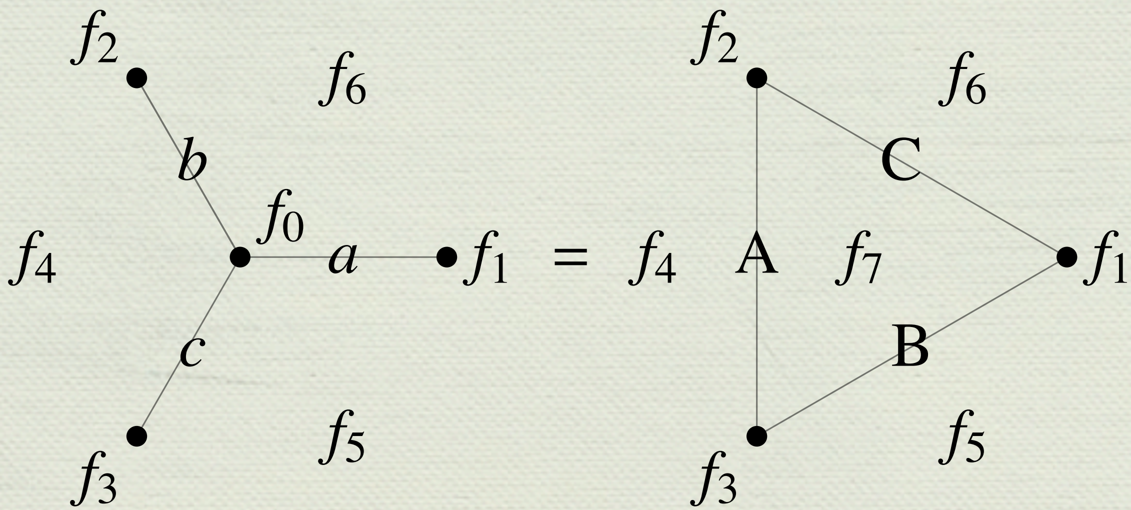
$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}}$$

$$B = \sqrt{\frac{(abc+1)(b+ac)}{(a+bc)(c+ab)}}$$

$$C = \sqrt{\frac{(abc+1)(c+ab)}{(a+bc)(b+ac)}}$$

Remarkable fact about the Ising Y-Delta move (Kashaev):

Define new variables f on vertices and faces: "activities"



The activities are related to edge weights as:

$$\left(\frac{a-1/a}{2}\right)^2 = \frac{f_0 f_1}{f_5 f_6}, \qquad etc.$$

Theorem [Kashaev] The fs satisfy

$$f_0^2 f_7^2 + f_1^2 f_4^2 + f_2^2 f_5^2 + f_3^2 f_6^2 - 2(f_1 f_2 f_4 f_5 + f_1 f_4 f_3 f_6 + f_2 f_3 f_5 f_6)$$
$$-2 f_0 f_7 (f_1 f_4 + f_2 f_5 + f_3 f_6) - 4(f_0 f_4 f_5 f_6 + f_7 f_1 f_2 f_3) = 0.$$

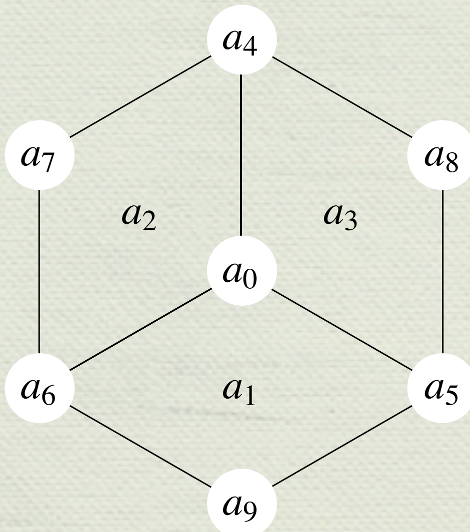
(This is the algebraic identity satisfied by the principal minors of a 3×3 symmetric matrix.)

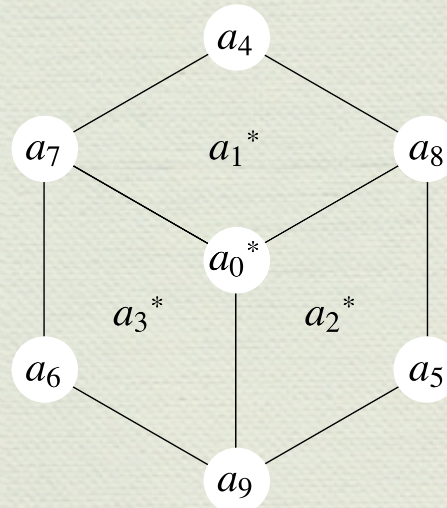
We say $f : \mathbb{Z}^3 \to \mathbb{C}$ **satisfies the Kashaev recurrence**

if $P(f_{i,j,k}, f_{i+1,j,k}, \ldots, f_{i+1,j+1,k+1}) = 0$ for all $(i, j, k) \in \mathbb{Z}^3$.

By defining $f_{i,j,k}$ on $0 \leq i + j + k \leq 2$ we can use P to define it everywhere.

The garnet recurrence





$$
\begin{aligned}
a_1^* &= \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_4a_7}{a_0a_1} \\
a_2^* &= \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_5a_8}{a_0a_2} \\
a_3^* &= \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_6a_9}{a_0a_3} \\
a_0^* &= \frac{a_1^2a_2^2a_3^2 + a_1a_2a_3(2a_4a_5a_6 + a_0a_4a_7 + a_0a_5a_8 + a_0a_6a_9) + (a_5a_6 + a_0a_7)(a_4a_5 + a_0a_9)(a_4a_6 + a_0a_8)}{a_0^2a_1a_2a_3}.
\end{aligned}
$$

Main results:

Theorem [Propp, FZ, KP]:
The octahedron, cube and garnet recurrences are compositions of urban renewal transformations.

Corollary [FZ]: Laurent phenomenon.

Theorem [KP]:

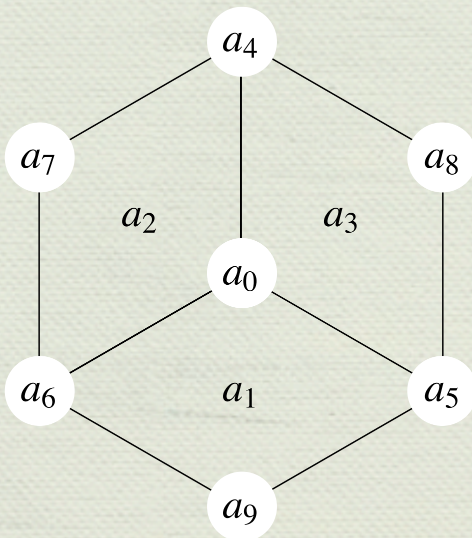
The Kashaev recurrence is a special case of the garnet recurrence.

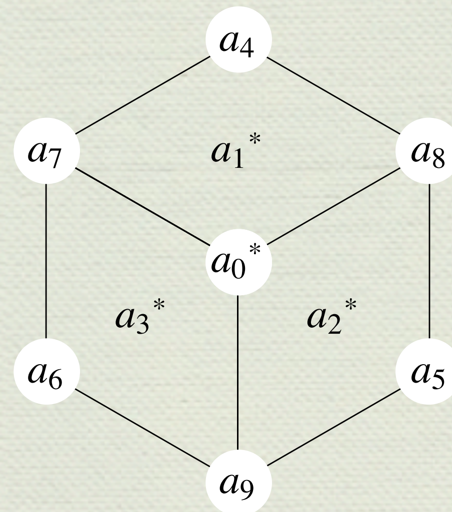
Corollary [KP]: Laurent phenomenon.

Open question:

Identify monomials of Kashaev recurrence with some combinatorial objects.

The Kashaev recurrence as a special case of the garnet recurrence.



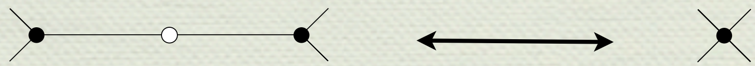


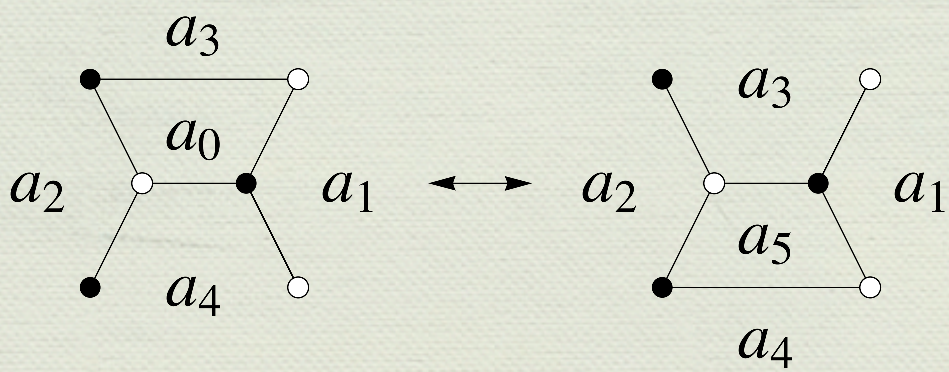
$$
\begin{aligned}
a_1 &= \sqrt{a_0 a_9 + a_5 a_6} \\
a_2 &= \sqrt{a_0 a_7 + a_4 a_6} \\
a_3 &= \sqrt{a_0 a_8 + a_4 a_5}
\end{aligned}
$$

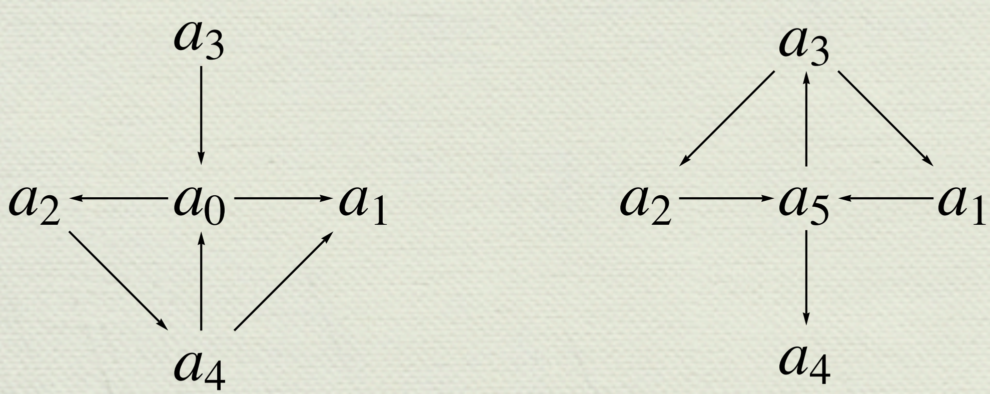
$$
\begin{aligned}
a_1^* &= \sqrt{a_0^* a_4 + a_7 a_8} \\
a_2^* &= \sqrt{a_0^* a_5 + a_8 a_9} \\
a_3^* &= \sqrt{a_0^* a_6 + a_7 a_9}
\end{aligned}
$$

Urban renewal variant

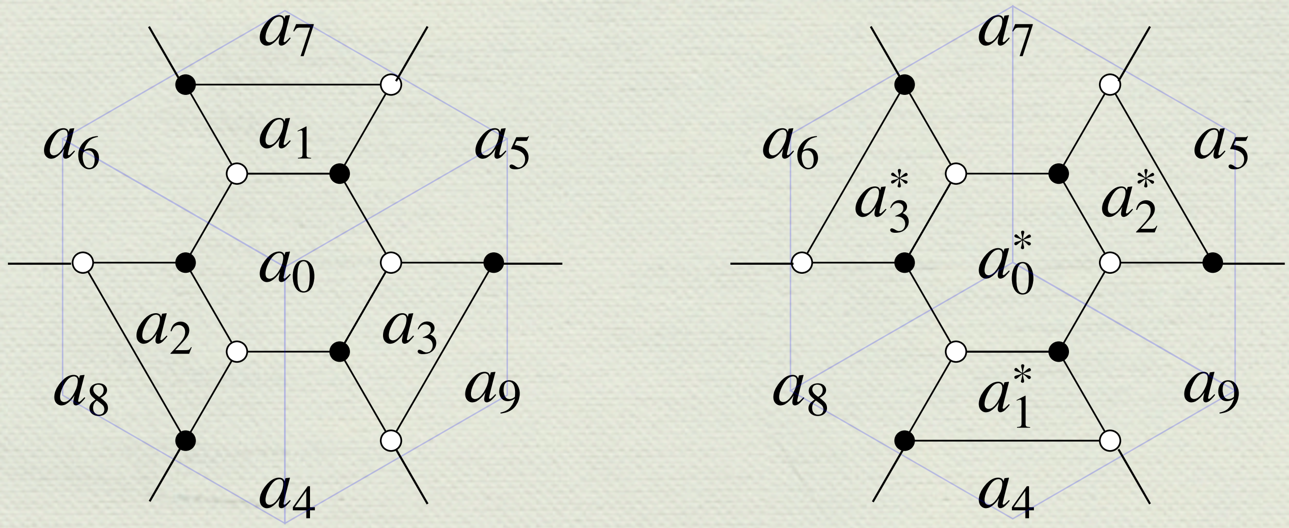
(equivalent under contraction/expansion of degree 2 vertices)

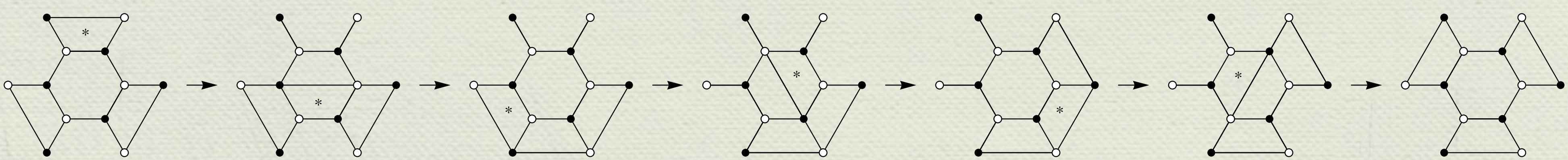






The garnet recurrence ("superurban renewal")

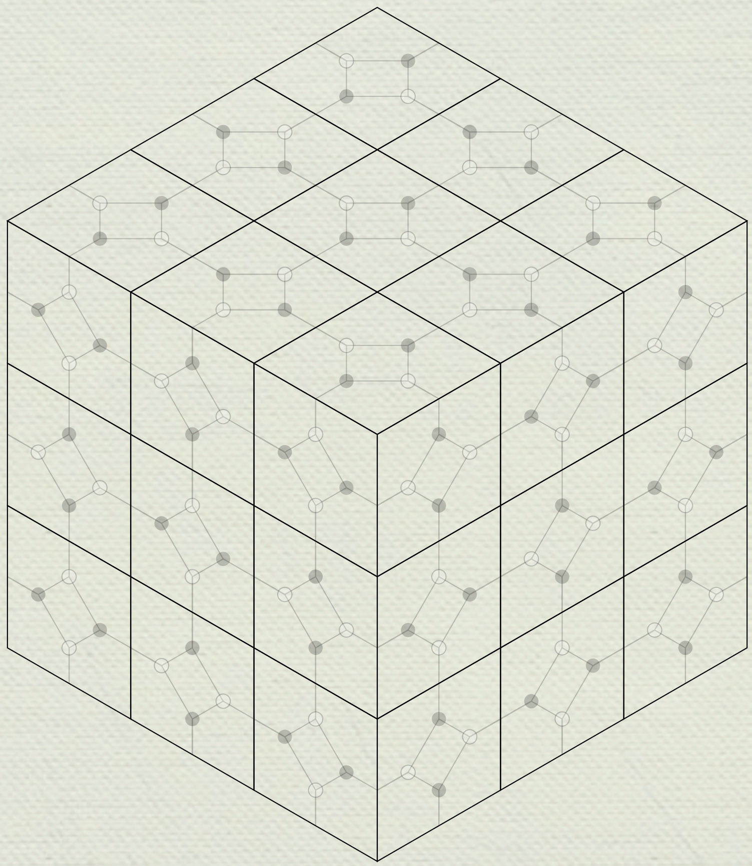




...is a composition of six urban renewals

What do the terms in the garnet recurrence count?

Certain “double-dimer” configurations on the cubic corner graph:



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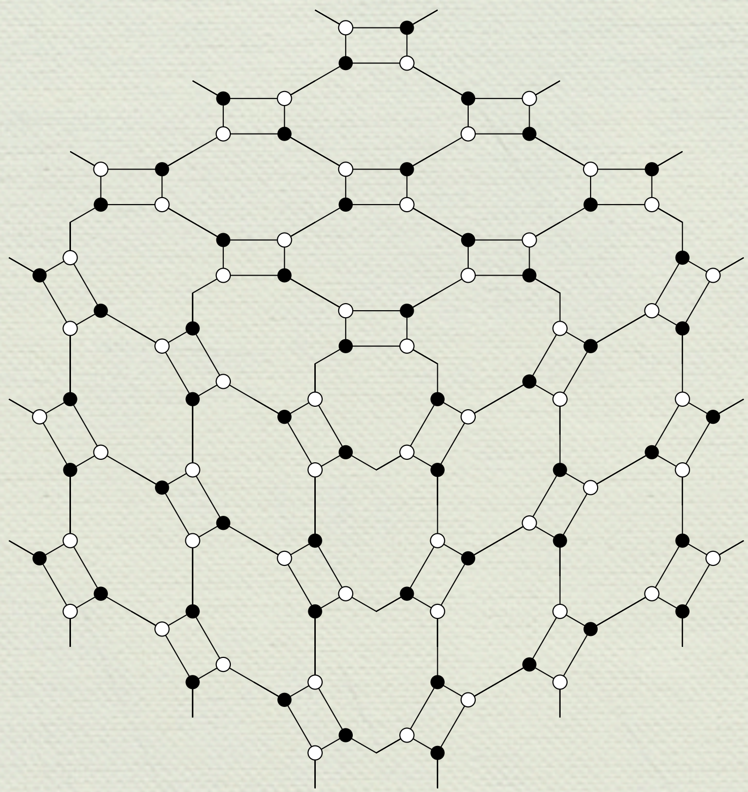
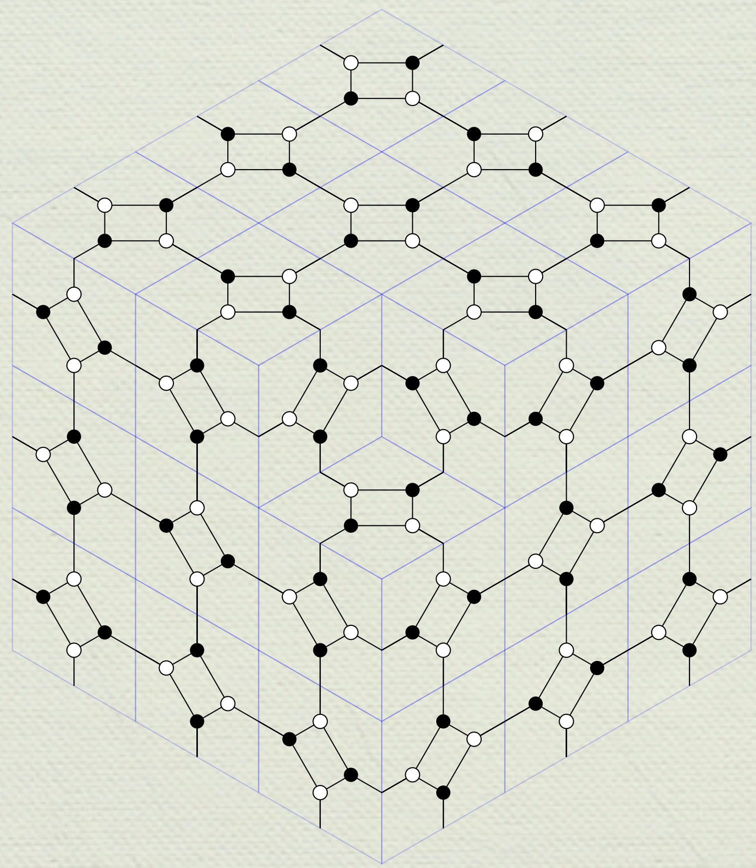
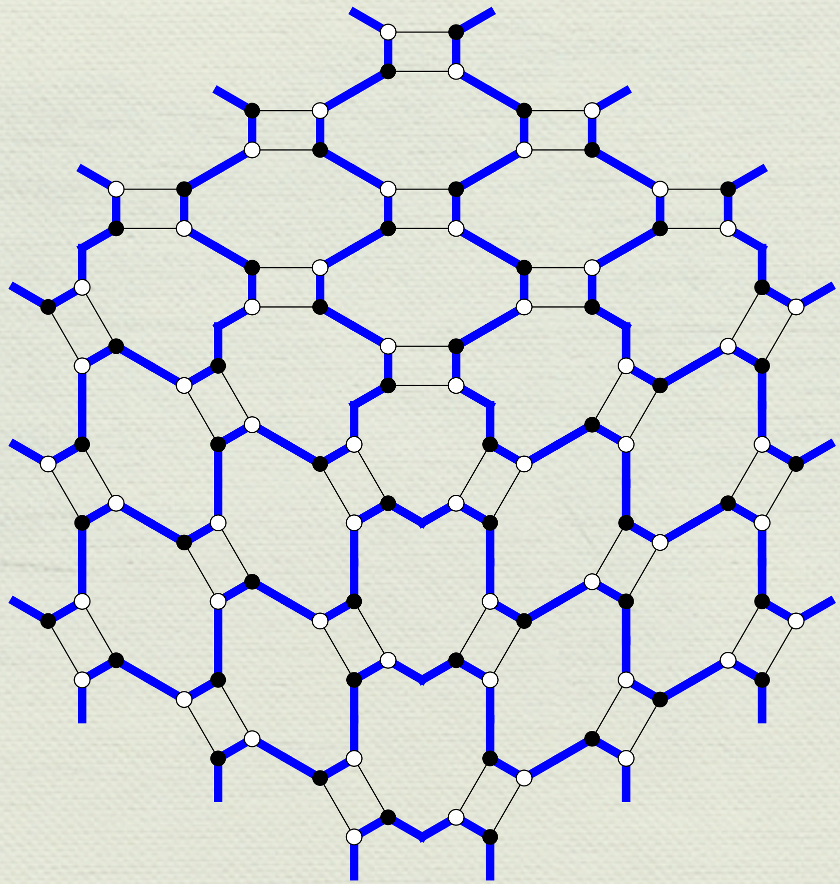


Γ_n is obtained by "removing" cubes $x + y + z < n$.

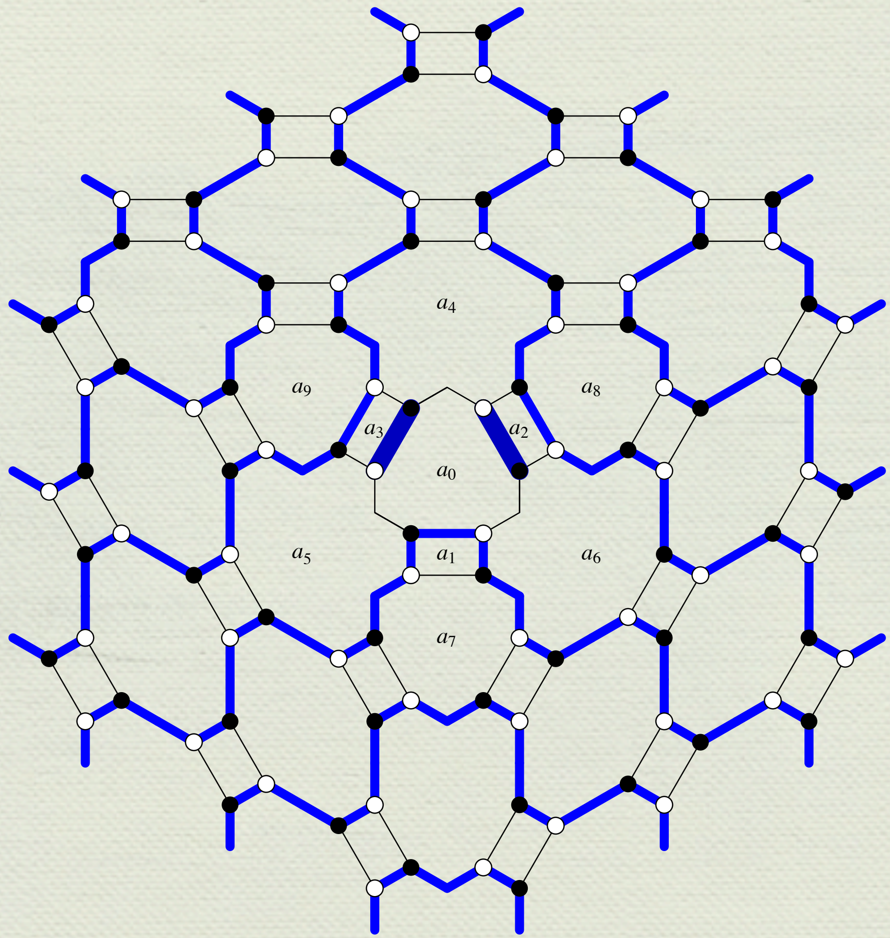
Γ_1 :





The initial configuration m_0 on Γ_0.

When you “remove” a cube the configuration changes *locally* (and randomly) and preserving topology.

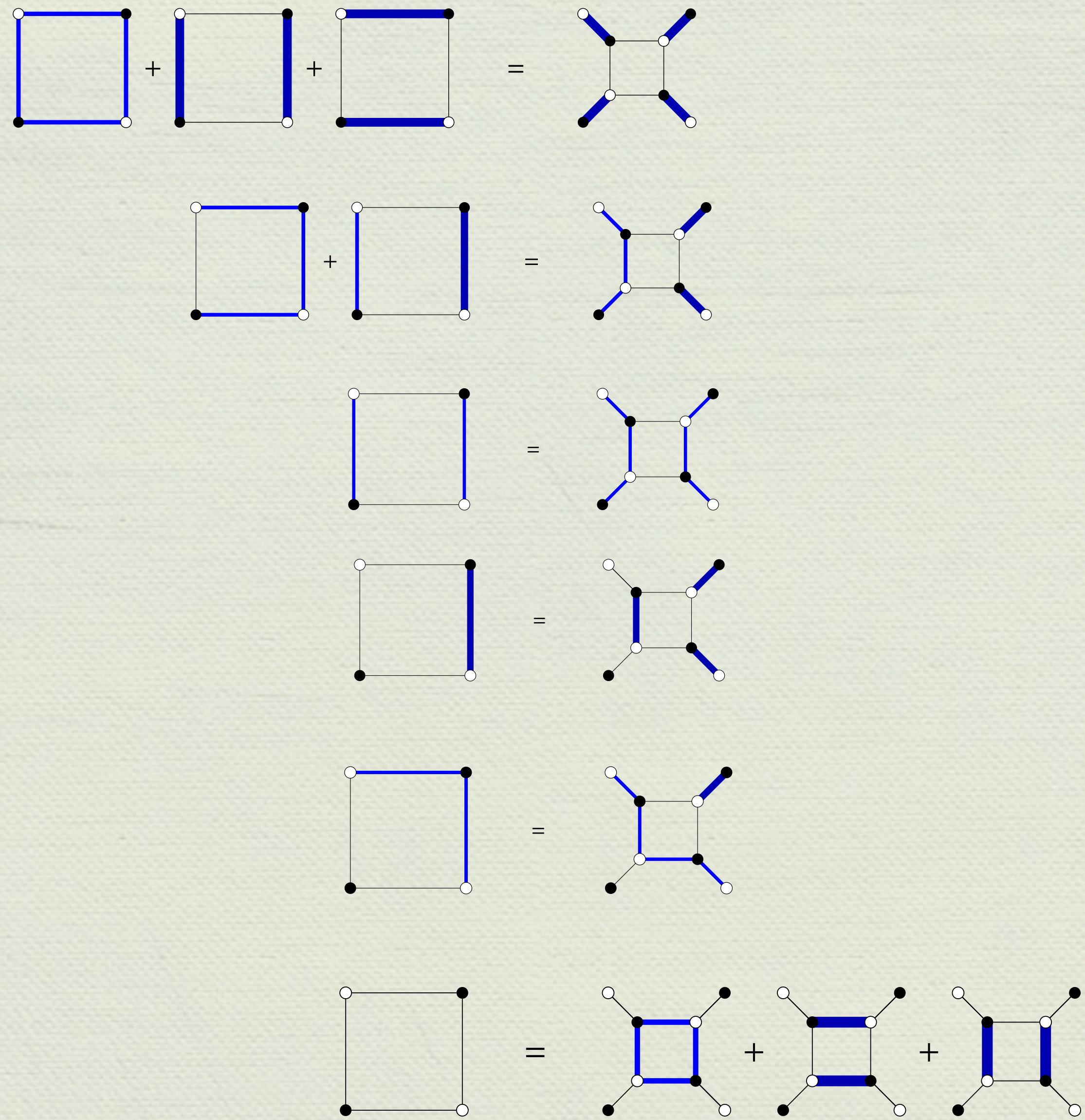


weight $= \dfrac{a_4^2 a_5 a_6 a_7}{a_0 a_1 a_2 a_3}$

A taut configuration on Γ_1. (Same connections as m_0).

Thm: $a_{i,i,i}$ is a Laurent polynomial whose monomials are in bijection with "taut" double-dimer covers of Γ_i.

Proof: Check that urban renewal preserves topology...

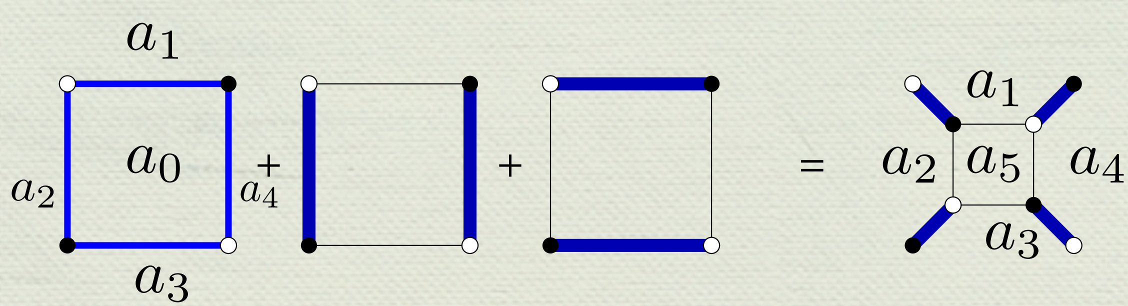


magic formula:

number of loops; length of face

the monomial for a configuration is $$2^c \prod_{\text{faces } f} a_f^{L-2-\#edges}.$$

for example:



$$\frac{2}{a_0^2 a_1 a_2 a_3 a_4} + \frac{1}{a_0^2 a_2^2 a_4^2} + \frac{1}{a_0^2 a_1^2 a_3^2} = \frac{a_5^2}{a_1^2 a_2^2 a_3^2 a_4^2}$$

Arctic circle theorem: use $a_{i,j,k} = 3^{(i+j+k)^2/2}$ (vertex variables)

$a_{i,j,k} = 2 \cdot 3^{(i+j+k+1)^2/2}$ (face variables)

then $\frac{a'_{i,j,k}}{a_{i,j,k}}$ satisfies a linear recurrence (with constant coefficients).

Let $G(x,y,z) = \sum \frac{a'_{i,j,k}}{a_{i,j,k}} x^i y^j z^k$.

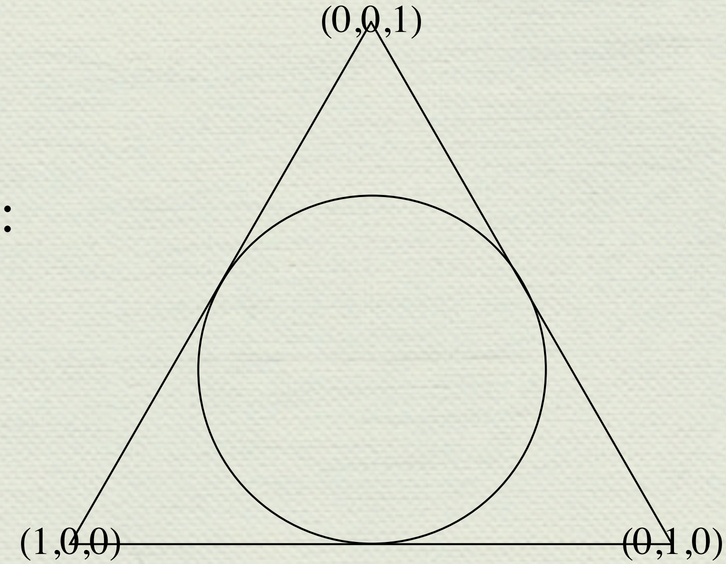
Then $G(x,y,z)$ satisfies a linear recurrence with characteristic polynomial:

$$P(x,y,z) = xyz + 1 - \tfrac{1}{3}(xy + xz + yz + x + y + z).$$

Analyze growth of coefficients of $1/P$:

- polynomial inside inscribed circle
- exponential decay outside inscribed circle QED.

Specific "3^n" initial conditions:



More generic initial conditions