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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SKHERMES@BRANDEIS.EDU

Speaker's Name: P. Di Francesco

Talk Title: Periodicity, positivity, & Integrability of T-systems

Date: 10 / 30 / 12 **Time:** 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: T-systems, Laurent positivity, periodicity, Yang-Baxter Equation, Mutation, Discrete Integrable Systems

Please summarize the lecture in 5 or fewer sentances: The speaker introduced a family of discrete integrable systems called T-systems which embed into cluster algebras. He shows that the solutions are a positive Laurent polynomial in the choice of initial data for several classes of T-systems.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

PERIODICITY, POSITIVITY & INTEGRABILITY OF T-SYSTEMS

(P. DiFrancesco & R. Kedem)

[arXiv:1002.4427; 1208.4333]

PERIODICITY, POSITIVITY & INTEGRABILITY OF T-SYSTEMS

(P. DiFrancesco & R. Kedem)

Discrete Integrable Systems

- evolution eqns in discrete time $k \in \mathbb{Z}$
- conservation laws
- dependence on initial data/boundary conditions
  - Laurent positivity [Cluster Algebras]
  - Periodicity [Zamolodchikov '90s]
  - Network Exact solutions [DF-K]

T SYSTEM

general form : (A_∞, discrete Hirota, octahedron)

$$\boxed{T_{\alpha\, j\, k+1}\, T_{\alpha\, j\, k-1} = T_{\alpha\, j+1\, k}\, T_{\alpha\, j-1\, k} + T_{\alpha+1, j, k}\, T_{\alpha-1, j, k}}$$

$$\alpha, j, k \in \mathbb{Z}$$

α: reps index

j: spectral parameter

k: discrete time

- Fusion relations for transfer matrices of generalized Heisenberg quantum spin chains (type A)
- Frieze patterns
- LR rules
- λ-det, ASM.

Aztec diamond tilings

BOUNDARY CONDITIONS

- appeared first with $\alpha \geq 1$ condition (reps. index)

and

$$\boxed{T_{0,j,k} = 1 \qquad \forall\, j,k \in \mathbb{Z}} \qquad (A_{\infty/2})$$

- Sl_{r+1} spin chains $1 \leq \alpha \leq r$ restriction

$$\boxed{T_{0,j,k} = T_{r+1,j,k} = 1 \quad \forall j,k \in \mathbb{Z}} \qquad (A_r)$$

Ex: (A_2) $\alpha = 1,2$ $\quad T_{1,j,k} = R_{j,k} \quad T_{2,j,k} = S_{j,k}$

$$\boxed{\begin{cases} R_{j,k+1} R_{j,k-1} = R_{j+1,k} R_{j-1,k} + S_{j,k} \\ S_{j,k+1} S_{j,k-1} = S_{j+1,k} S_{j-1,k} + R_{j,k} \end{cases}} \qquad (A_2)$$

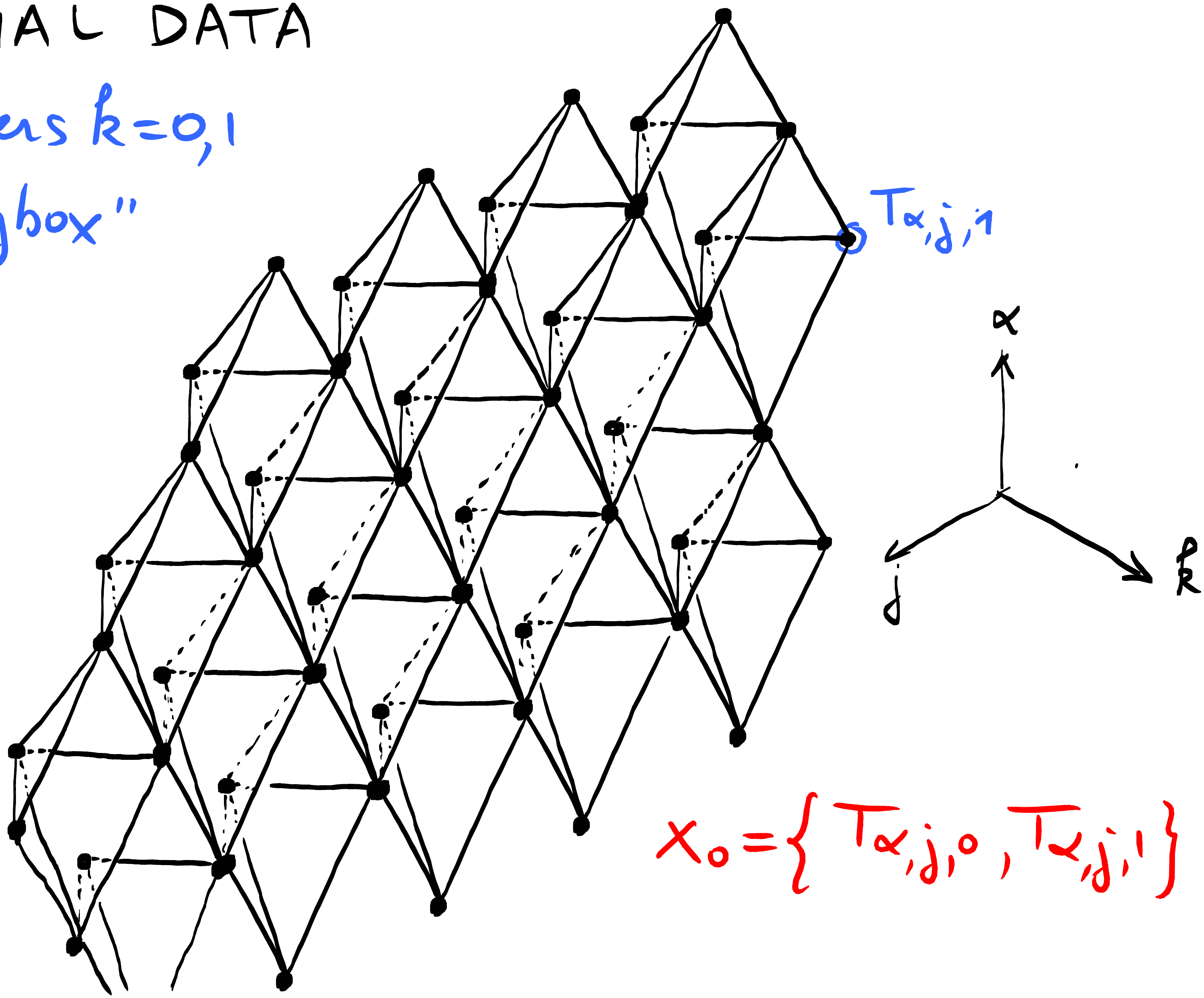
LAURENT POSITIVITY

- all these T-systems are embedded into Cluster algebras [Kedem 07 - DFK 09]. Positivity?
- A_∞ case: proved by [Speyer 04] for f^{al} initial data
- THM [DF-K] For various boundary conditions:

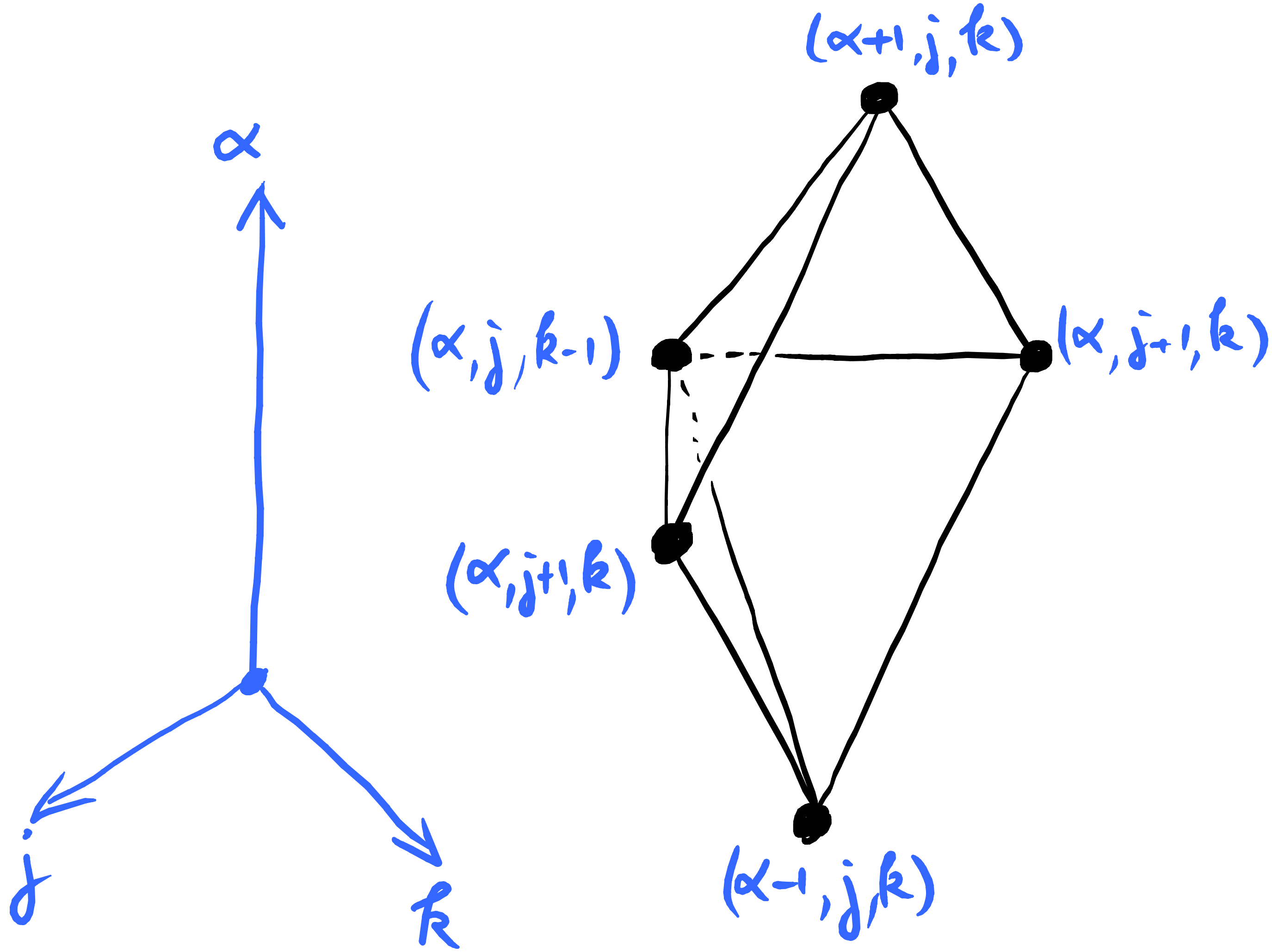
$$\boxed{\text{The Solution } T_{\alpha j k} \text{ of } T \text{ system is a } \geq 0 \text{ Laurent polynomial of any initial data}}$$

INITIAL DATA

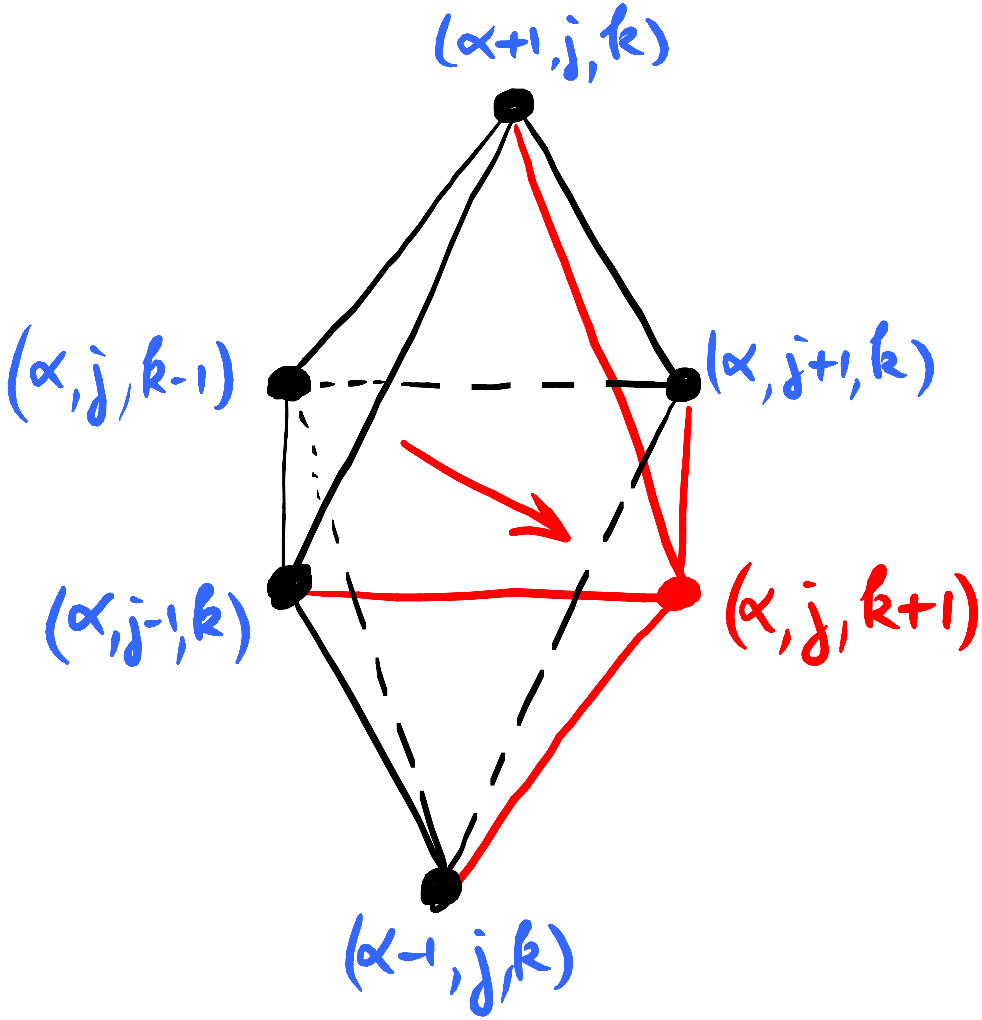
layers $k=0,1$ of "eggbox"



$X_0 = \{ T_{\alpha,j,0}, T_{\alpha,j,1} \}$



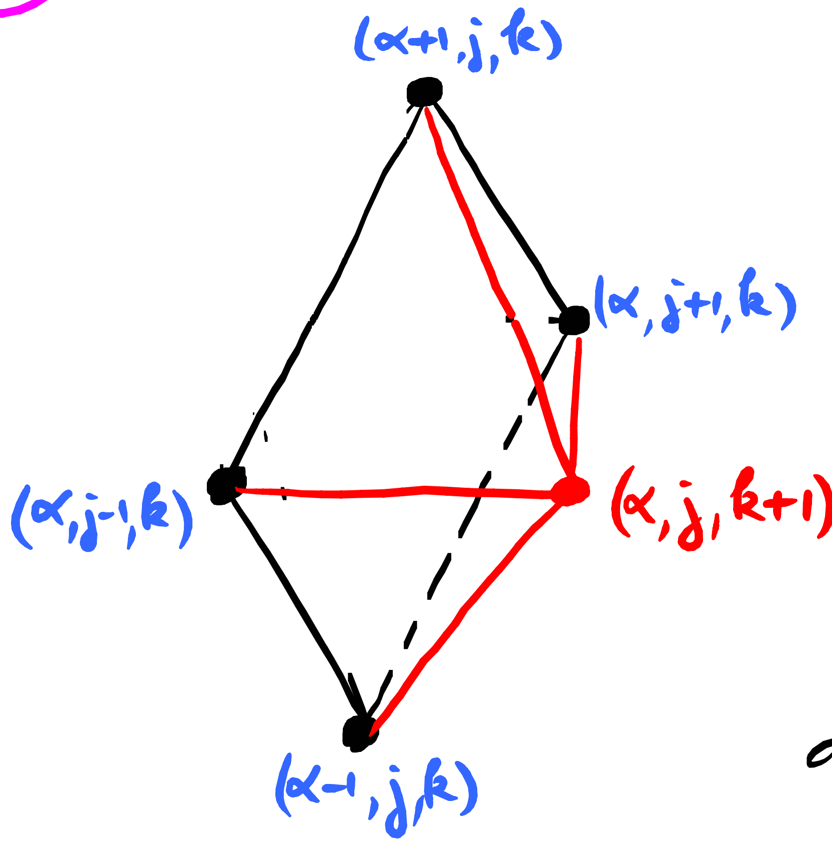
α
j
k
$(\alpha+1, j, k)$
$(\alpha, j, k-1)$
$(\alpha, j+1, k)$
$(\alpha, j+1, k)$
$(\alpha-1, j, k)$



(α+1, j, k)
(α, j, k-1)
(α, j+1, k)
(α, j-1, k)
(α, j, k+1)
(α-1, j, k)

MUTATION (α, j):

$$T_{\alpha j k+1}\, T_{\alpha j k-1} = T_{\alpha j+1 k}\, T_{\alpha j-1 k} + T_{\alpha+1, j, k}\, T_{\alpha-1, j, k}$$



local move that evolves the stepped surface by "adding" an octahedron at (α, j) and replacing the initial data $T_{\alpha j k-1}$ with $T_{\alpha j k+1}$

SUMMARY

$$\boxed{T_{\alpha j k+1} T_{\alpha j k-1} = T_{\alpha j+1 k} T_{\alpha j-1 k} + T_{\alpha+1, j, k} T_{\alpha-1, j, k}}$$

- initial data =

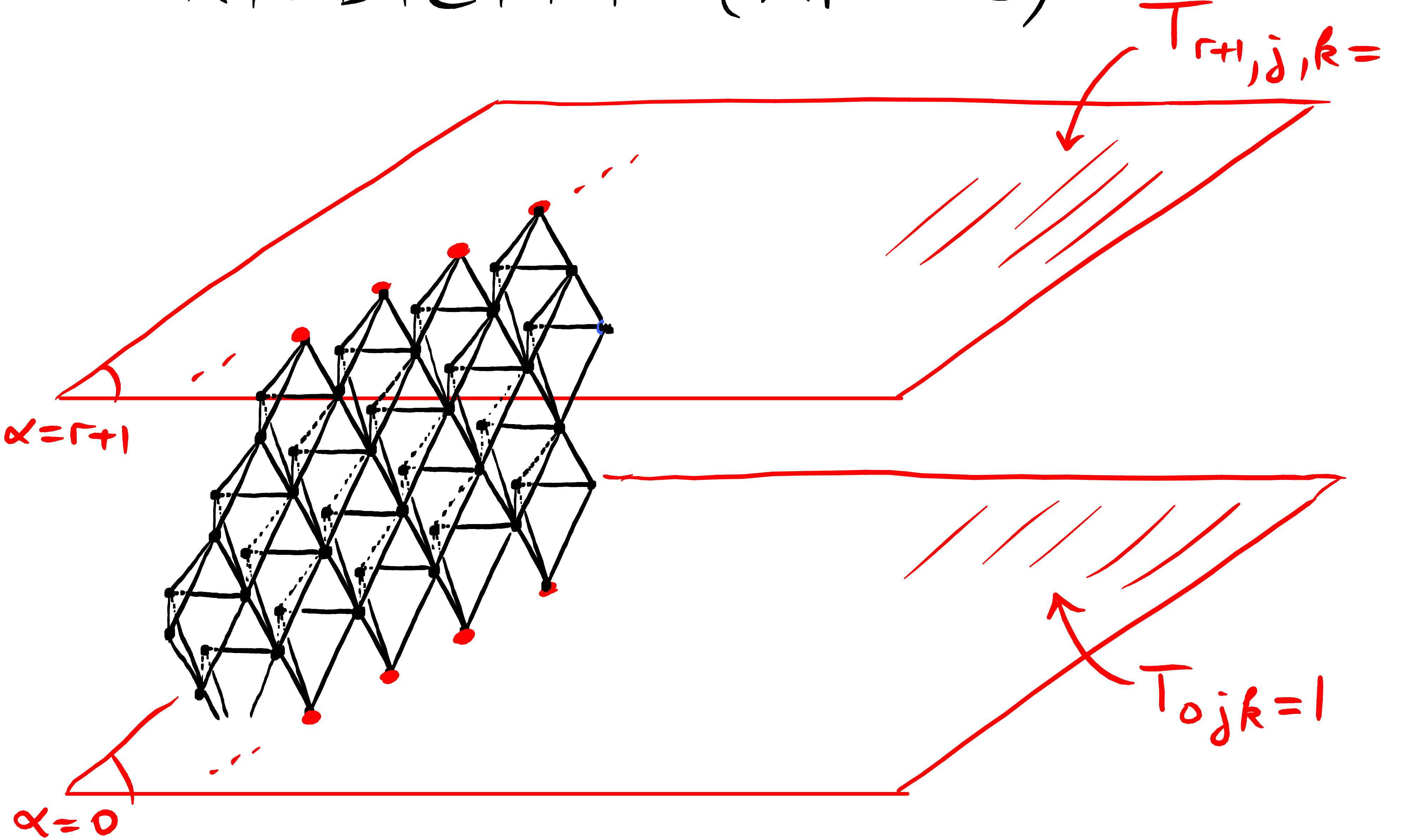
stepped surfaces $\underline{\underline{k}} = \{(\alpha, j, k_{\alpha, j})\}$ $\begin{cases} |k_{\alpha+1 j} - k_{\alpha j}| = 1 \\ |k_{\alpha j+1} - k_{\alpha j}| = 1 \end{cases}$

+ actual data $X_{\underline{\underline{k}}} = \{ T_{\alpha, j, k_{\alpha j}} \}$ = initial data

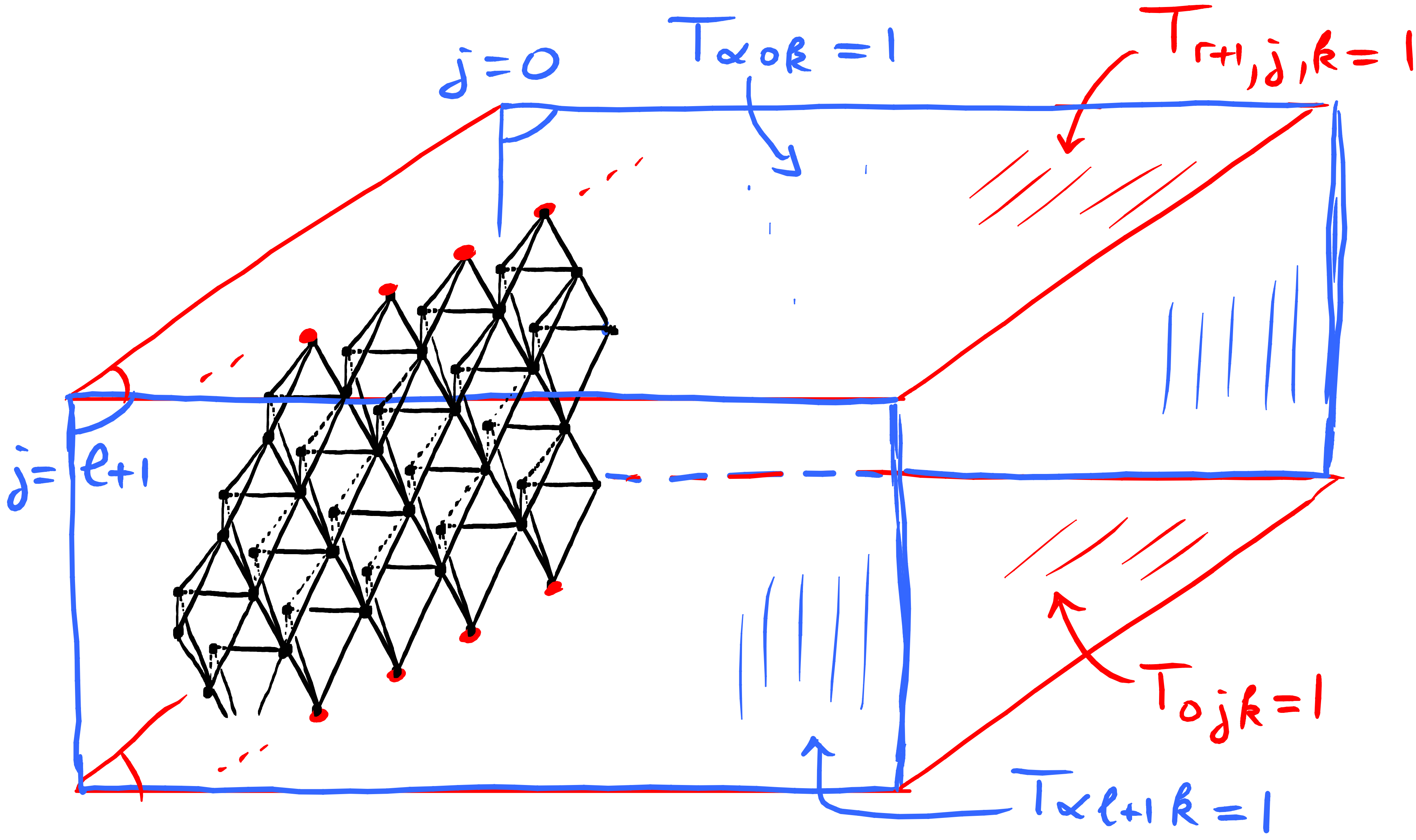
assignments along a stepped surface $\underline{\underline{k}}$

(NB: these are the clusters of the C.A.)

PERIODICITY (A_r case)



PERIODICITY (A_r case)



Zamolodchikov's periodicity conjecture (A case)

$$\begin{cases} T_{\alpha\, j\, k+1} T_{\alpha\, j\, k-1} = T_{\alpha\, j+1\, k} T_{\alpha\, j-1\, k} + T_{\alpha+1, j, k} T_{\alpha-1, j, k} & 1 \leq \alpha \leq r \\ & j, k \in \mathbb{Z} \\ T_{0 j k} = T_{r+1\, j k} = 1 & (A_r \text{ condition}) \\ T_{\alpha 0 k} = T_{\alpha\, \ell+1\, k} = 1 & (\text{Box of width } \ell) \end{cases}$$

THM The solution $T_{\alpha i j}$ is periodic, with period

$$\boxed{T = 2(r+\ell+2)}$$

Proofs: [Volkov] [Frenkel Szenes] [Keller] [Inoue et al] Today

T-SYSTEM SOLUTION (A_1 CASE)

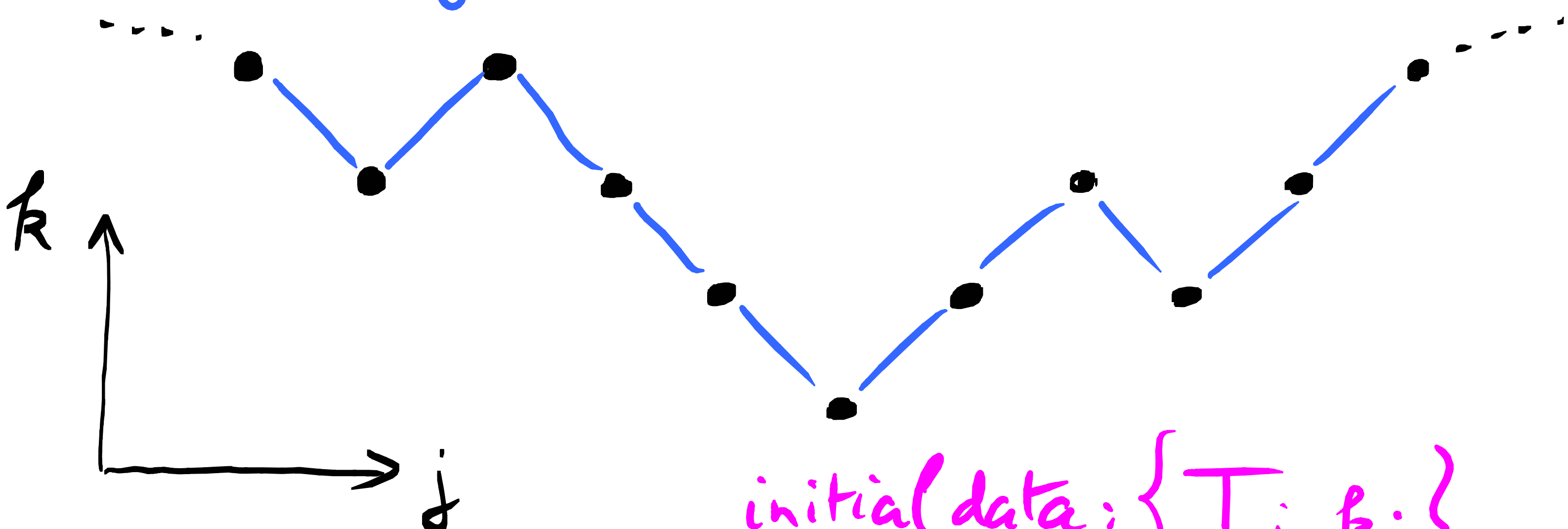
$$\boxed{T_{j,k+1}\,T_{j,k-1} = T_{j+1,k}\,T_{j-1,k} + 1}$$

$(\alpha=1)$

$j,k\in\mathbb{Z}$

initial data = zig-zag line $|k_{j+1}-k_j| = 1$

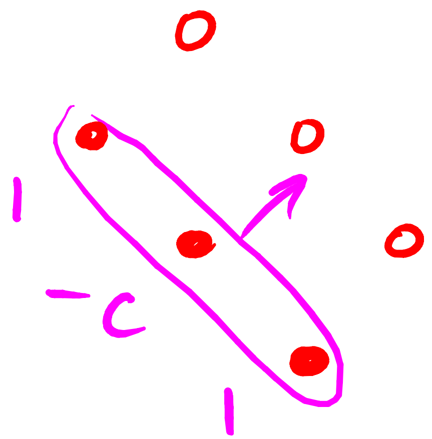
(one slice of A_r)



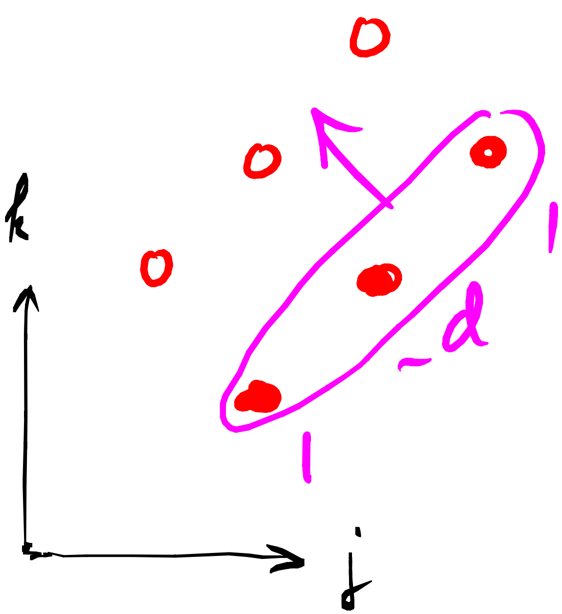
initial data: $\{T_{j,k_j}\}_{j\in\mathbb{Z}}$

INTEGRABILITY

2 types of conservation laws:

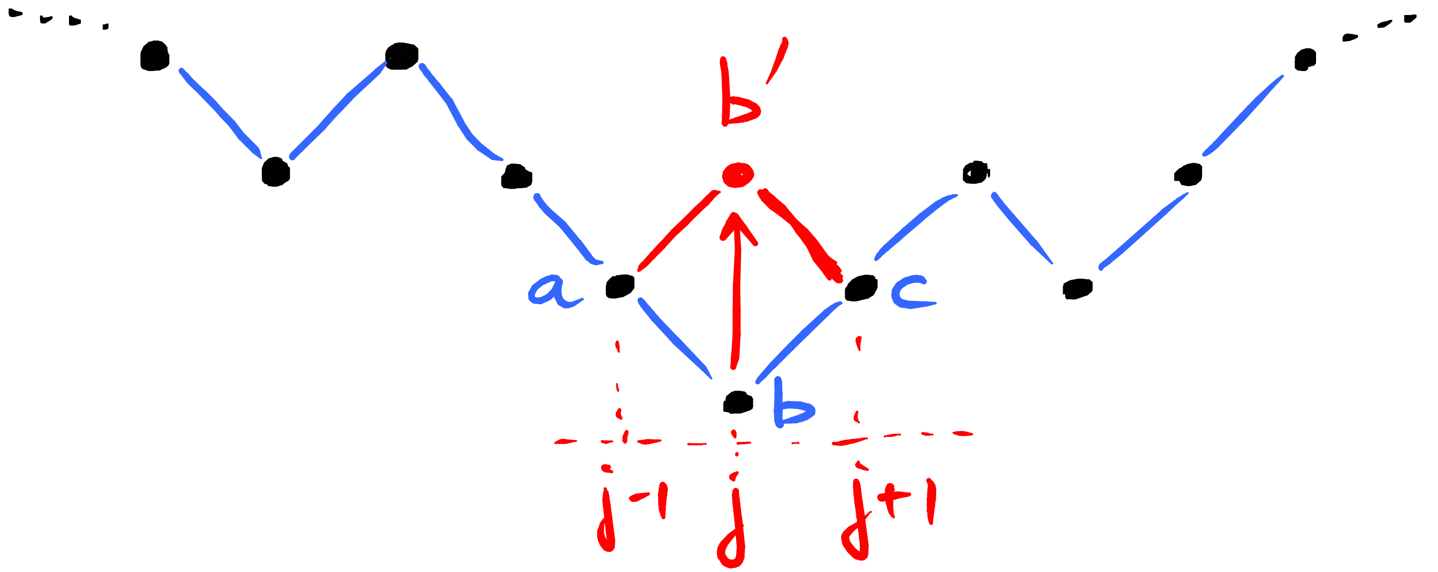


$$\boxed{T_{j,k+1} - c(j-k)\, T_{j+1,k} + T_{j+2,k-1} = 0}$$



$$\boxed{T_{j+2,k+1} - d(j+k)\, T_{j+1,k} + T_{j,k-1} = 0}$$

MUTATION = adding a "square"

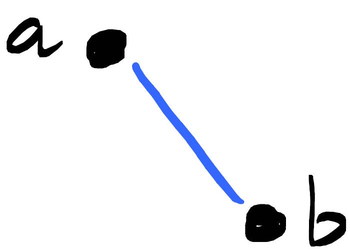


μ_j $\quad j \in \mathbb{Z}$

$$\boxed{b b' = a c + 1}$$

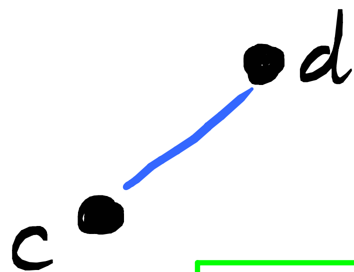
[NB inverse = subtracting a square]

MATRIX REPRESENTATION:



$$\longrightarrow \quad D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix}$$

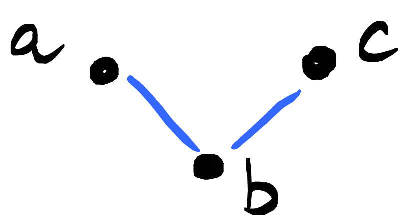
"down"

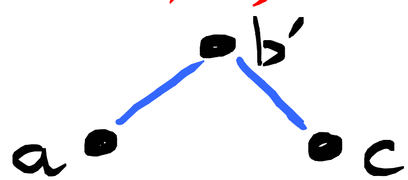


$$\longrightarrow \quad U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

"up"

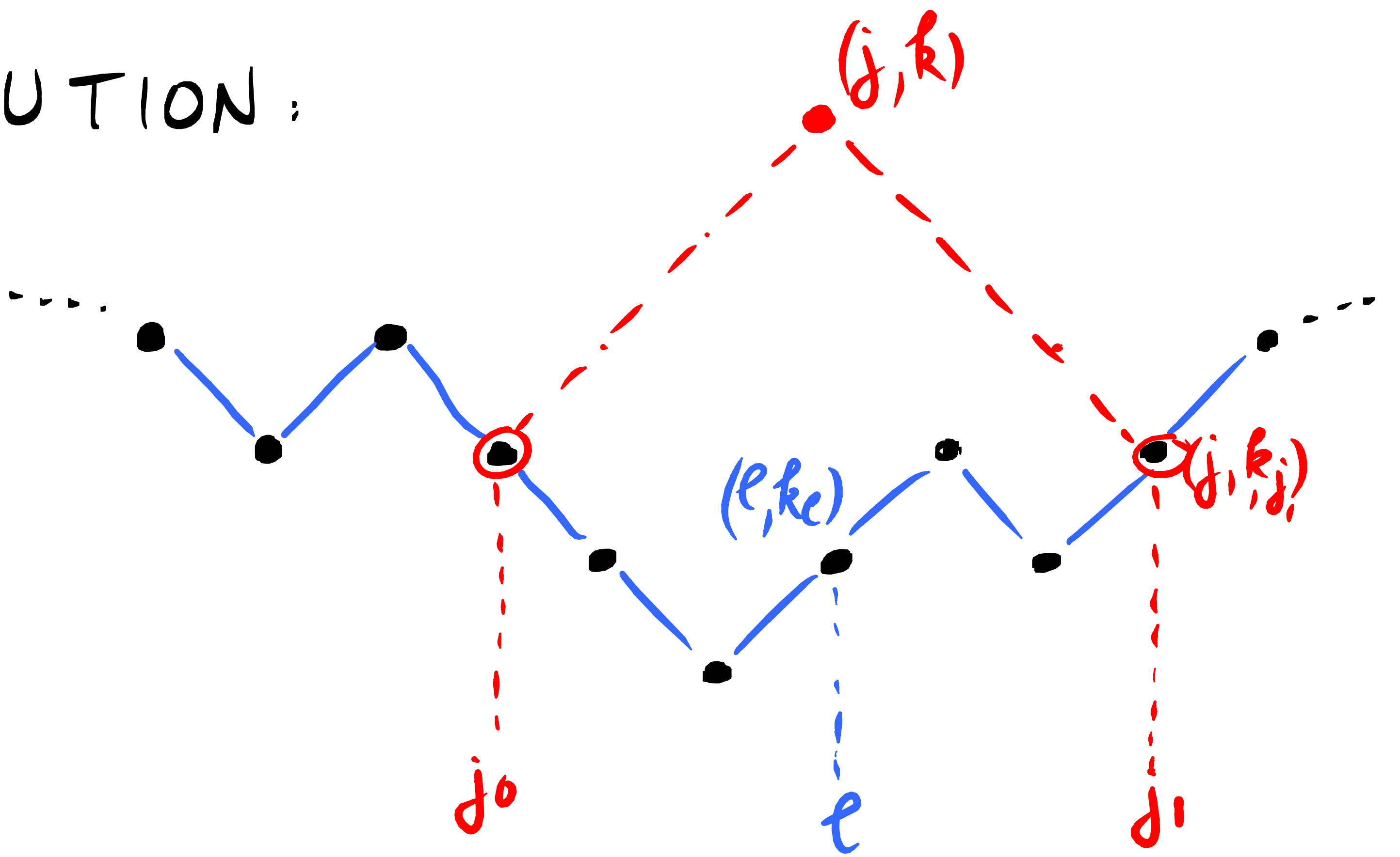
$$\boxed{D(a,b)\,U(b,c) = U(a,b')\,D(b',c) \iff bb' = ac+1}$$





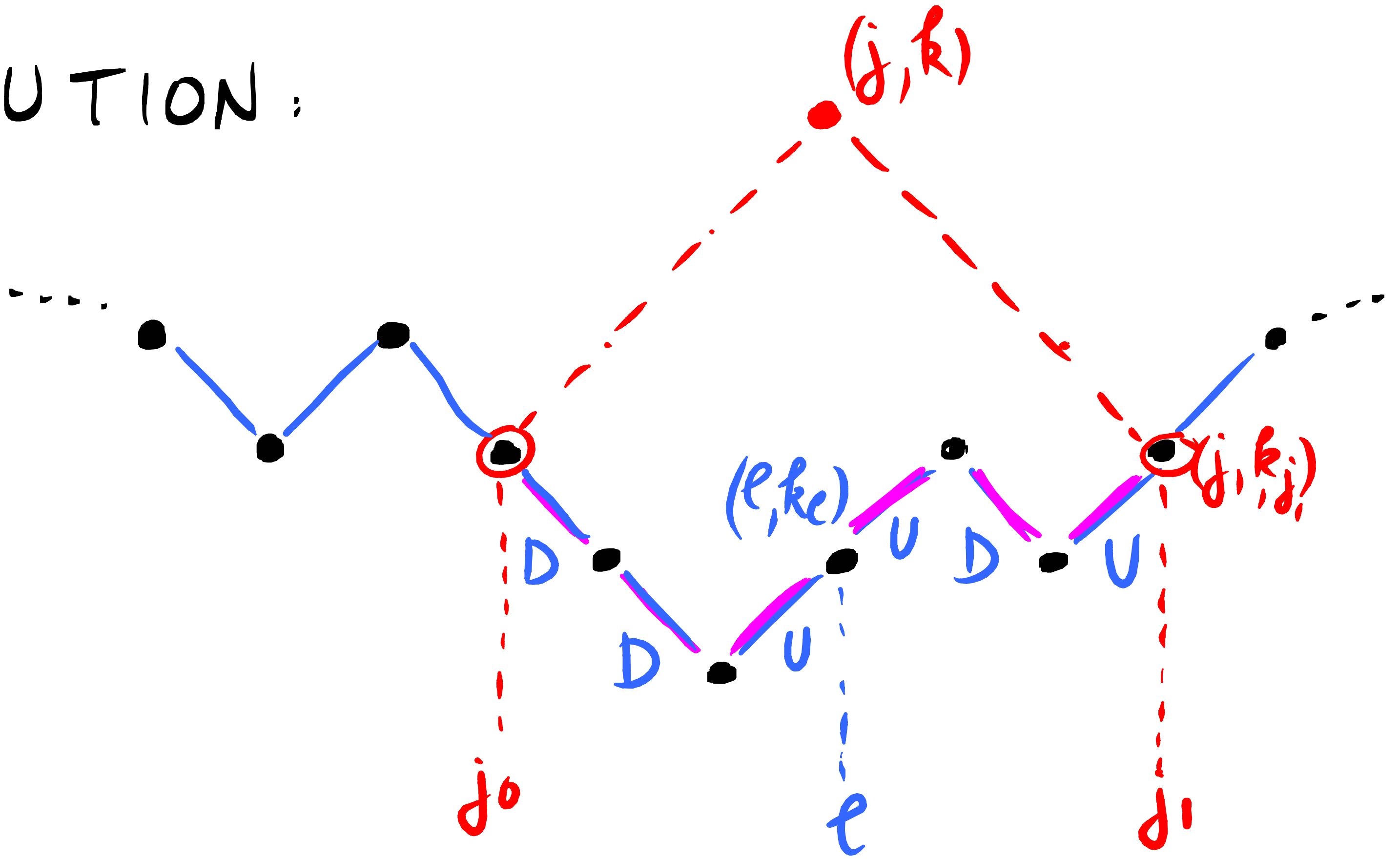
→ Implies the conservation laws (flat connexion)

SOLUTION:



$(j_0, j_1) = (\min, \max)$ of "projection" of (j,k) onto initial data zig-zag line. (j,k_j)

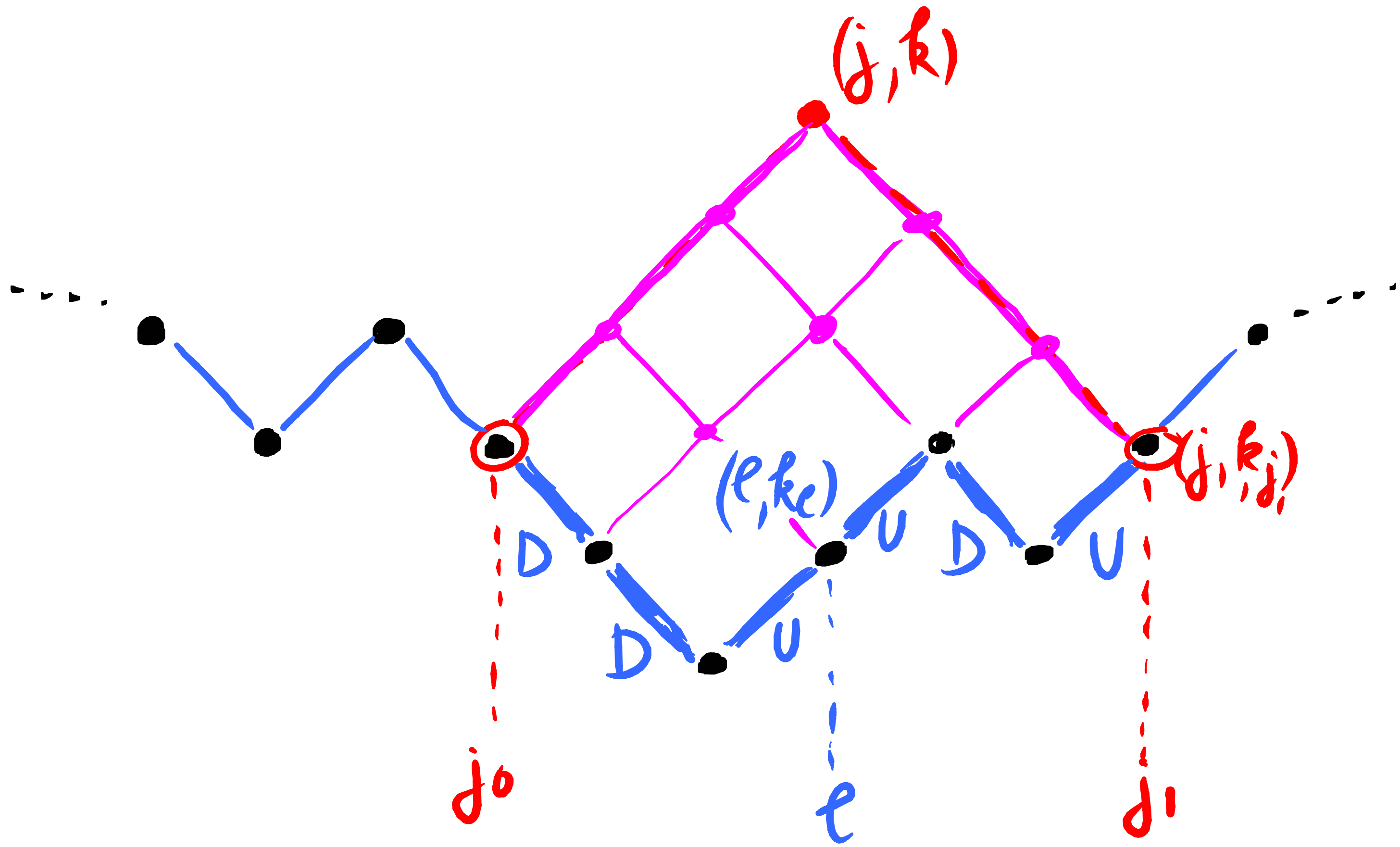
SOLUTION:



Thm $$\frac{T_{j,k}}{T_{j_1,k_{j_1}}} = \left[\prod_{\ell=j_0}^{j_1-1} \left\{ \begin{matrix} D \\ U \end{matrix} \right\} (k_\ell, k_{\ell+1}) \right]_{1,1} \leftarrow (1,1) \text{ element}$$

"Tranfer matrix"

Proof:



induction under mutation (box subtraction)

and $T_{j,k} = \left[(\prod U)(\prod D)\right]_{1,1} T_{j',k'}$

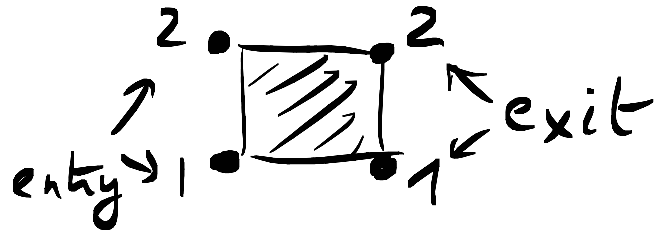
POSITIVITY

- the arguments of D, U involve only values of T_{j,k_j} from the initial data
- these are all ≥ 0 Laurent monomials

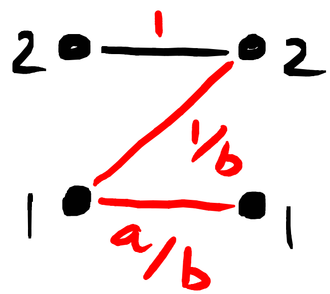
$\Rightarrow$ POSITIVITY

NETWORK FORMULATION

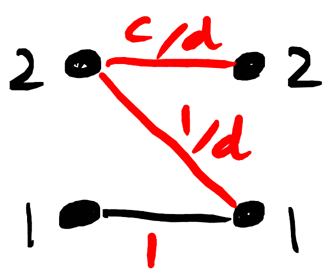
matrices ↔ weighted graphs (oriented left→right)



$$D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix} =$$

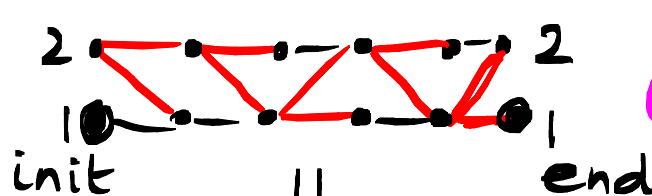


$$U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix} =$$



$$(U\ U\ D\ U\ D)_{1,1}$$

$\parallel$



$\parallel$

Product

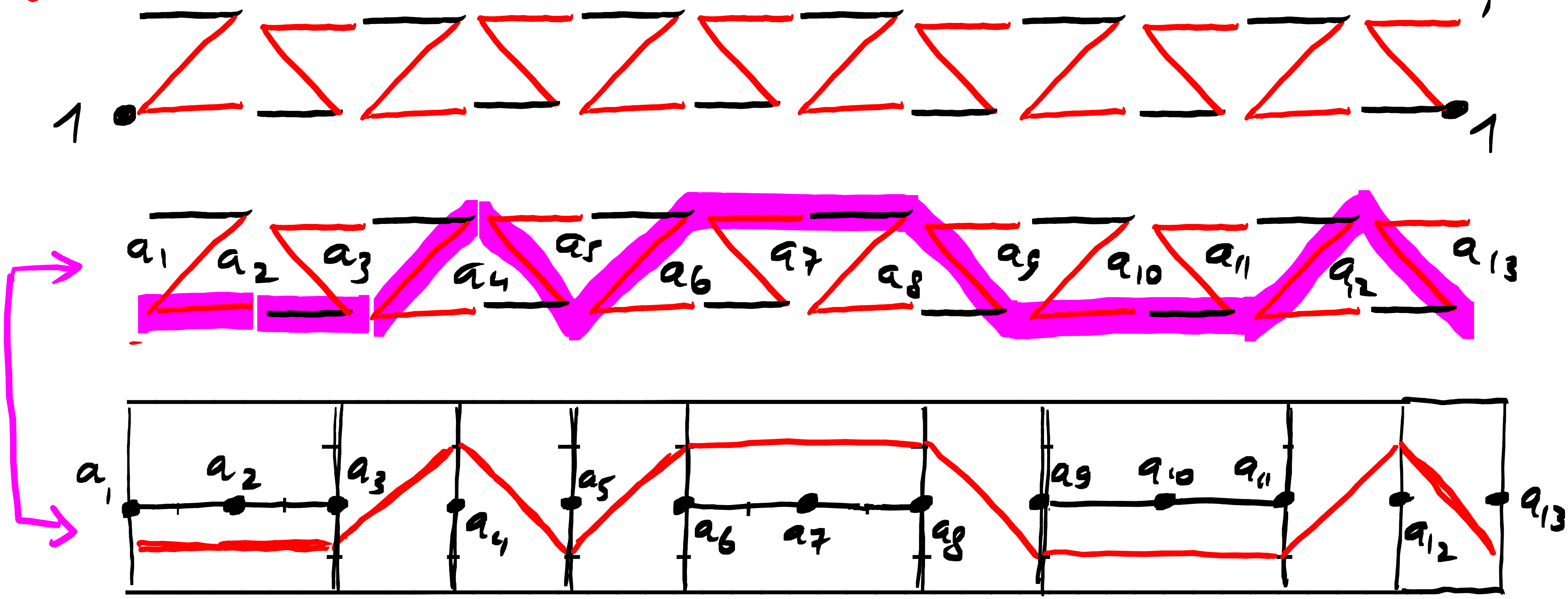
↓

Concatenate

= Partition function for paths $1 \to 1$ on the network.

CONNECTION TO DIMERS

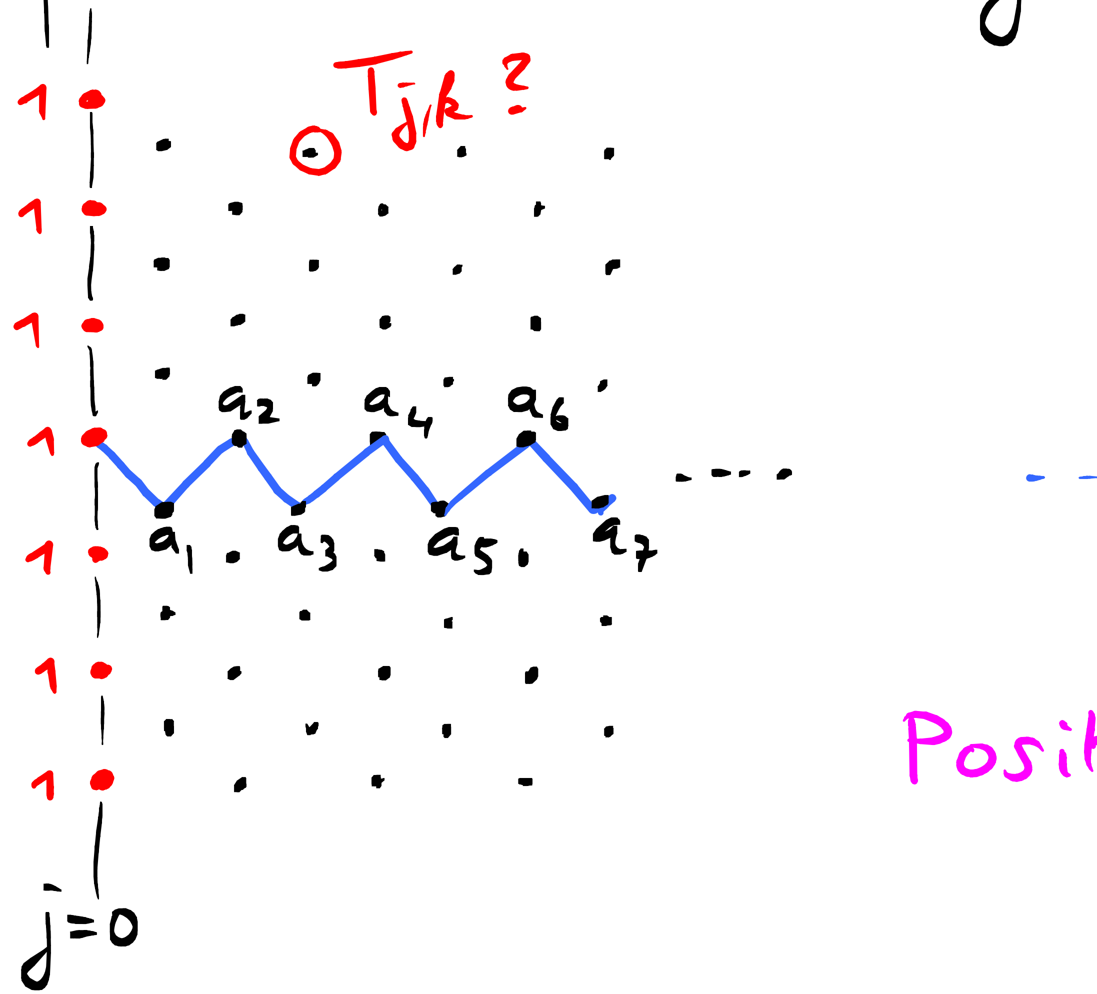
$$T_{jk} \sim \left(D_{a_1a_2}\, U_{a_2a_3}\, D_{a_3a_4}\, U\, D\, U\, U\, D\, U\, U\, D\, U\right)_{1,1}$$



bijection between:
- paths on network and
- between domino tilings + same weight

OTHER BOUNDARY CONDITIONS

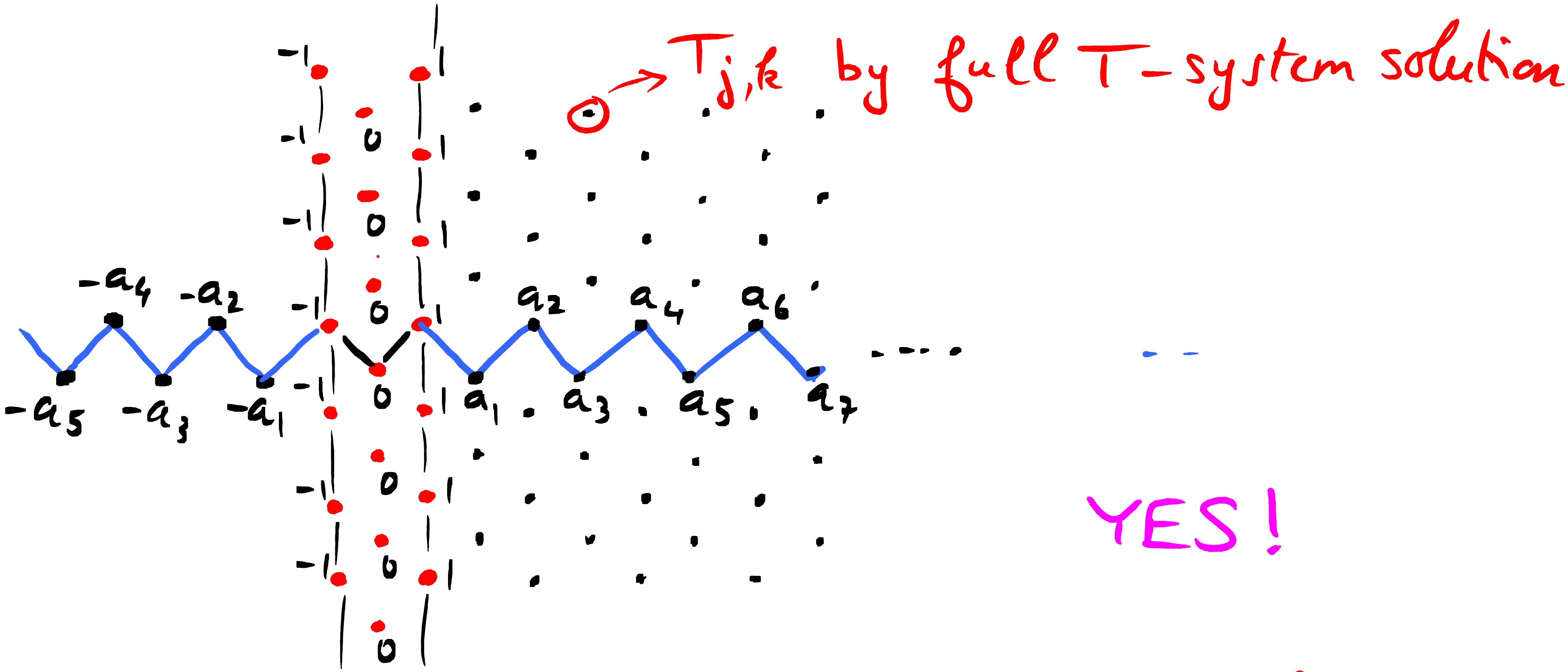
1. Half-plane "wall" boundary



Positivity ?

OTHER BOUNDARY CONDITIONS

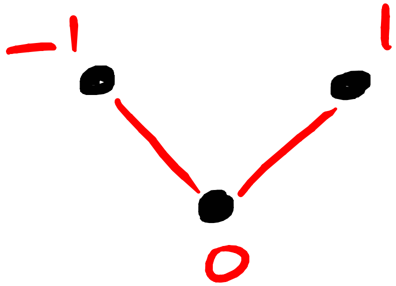
1. Half-plane "wall" boundary



Claim: the T-system with this init. data has same solution in the right half-plane

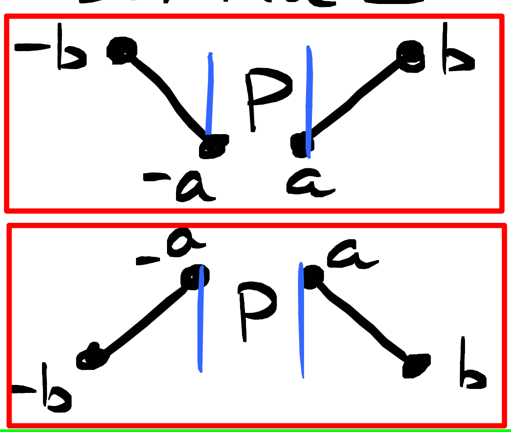
Properties of D, U, P matrices

Lemma 1



$$\lim_{\varepsilon \to 0} D(-1,\varepsilon)\, U(\varepsilon,1) = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Lemma 2



$$D(-b,-a)\, P\, U(a,b) = P$$

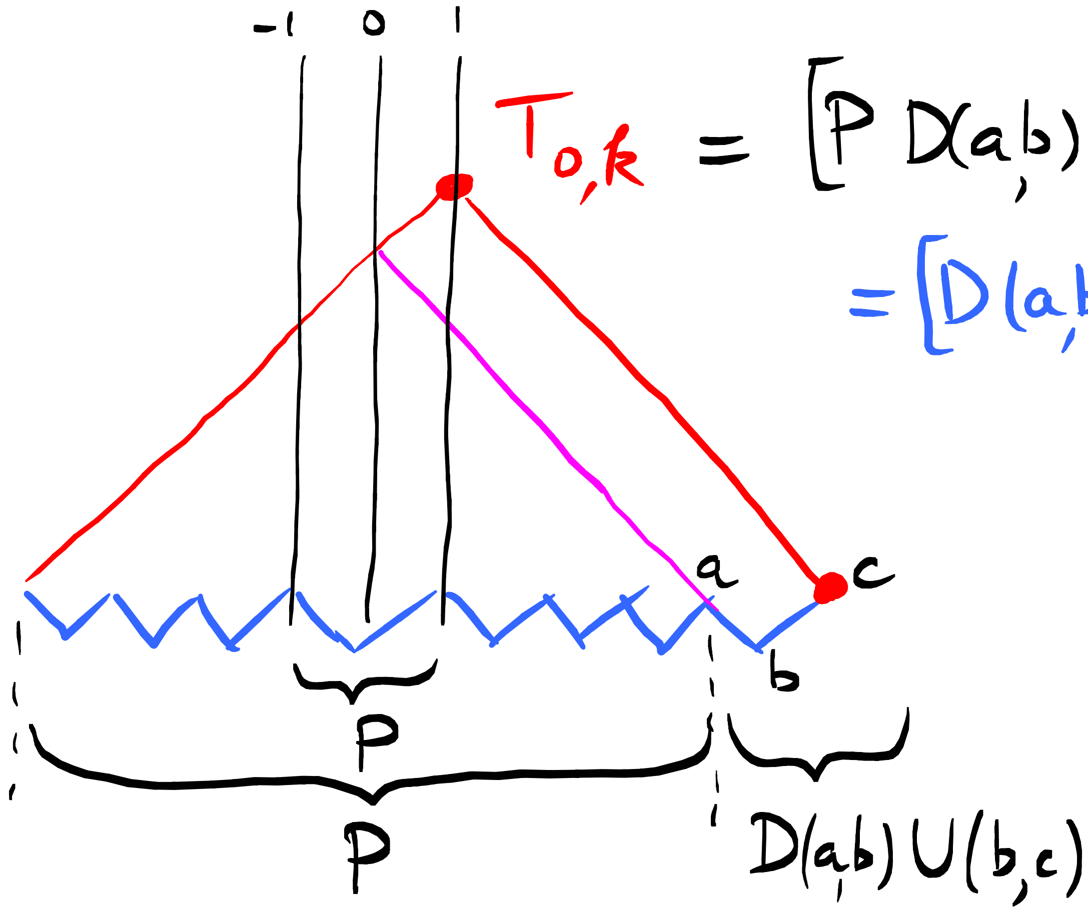
$$U(-b,-a)\, P\, D(a,b) = P$$

Corollary:

$= P$

COLLAPSE PROPERTY $\rho(\vec{a})$

Proof of the claim

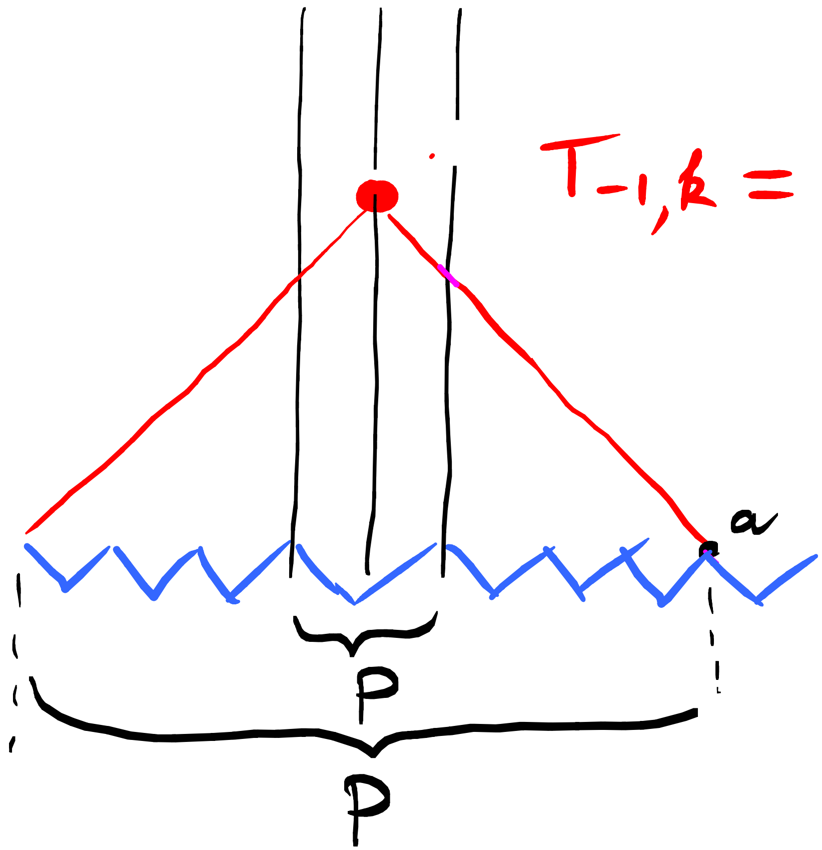


$$T_{0,k} = \left[P\, D(a,b)\, U(b,c) \right]_{1,1} \times c$$

$$= \left[D(a,b)\, U(b,c) \right]_{2,1} \times c$$

$$= 1$$

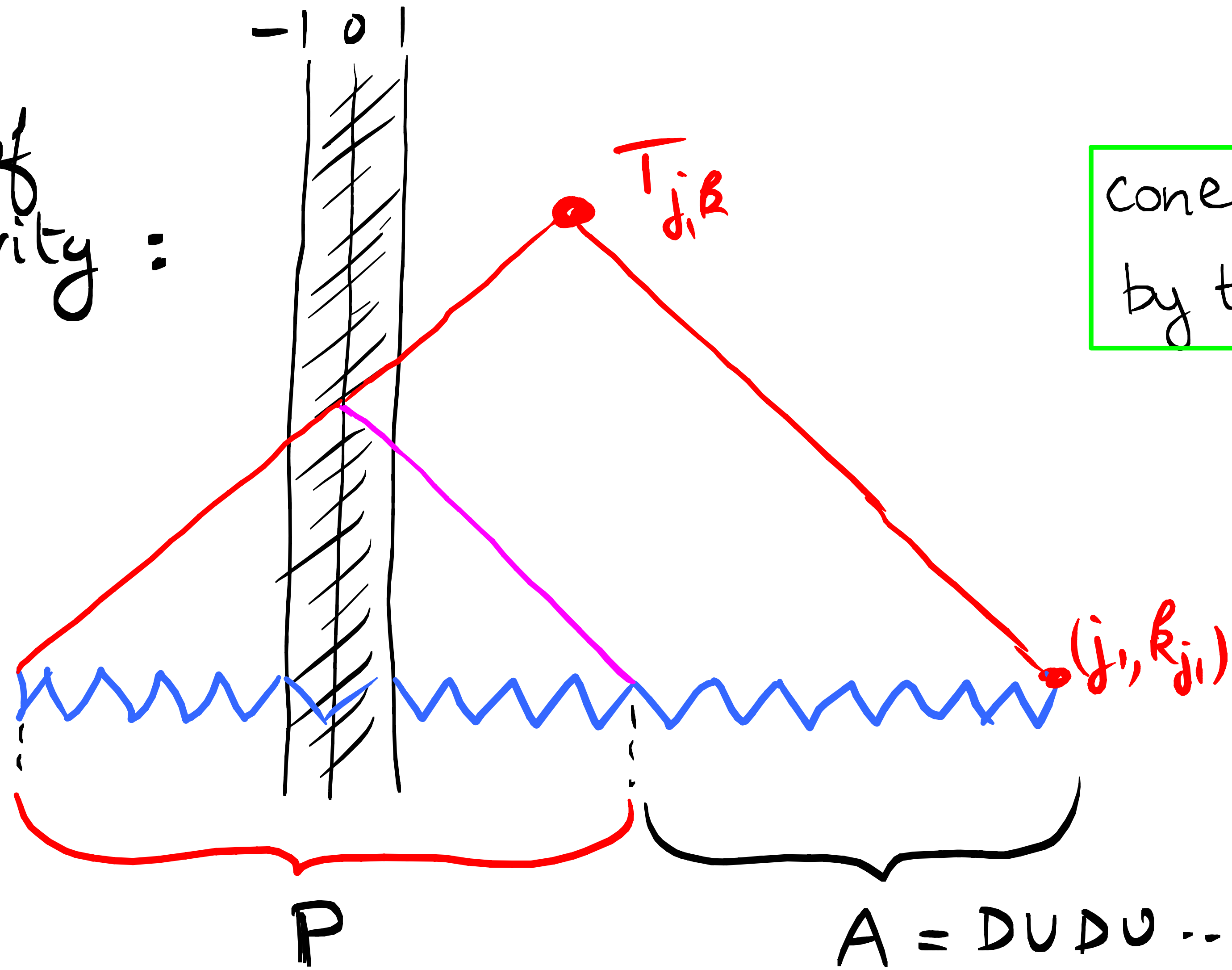
N.B.



$T_{-1,k} = [P]_{1,1} \times a = 0$

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Proof of Positivity :

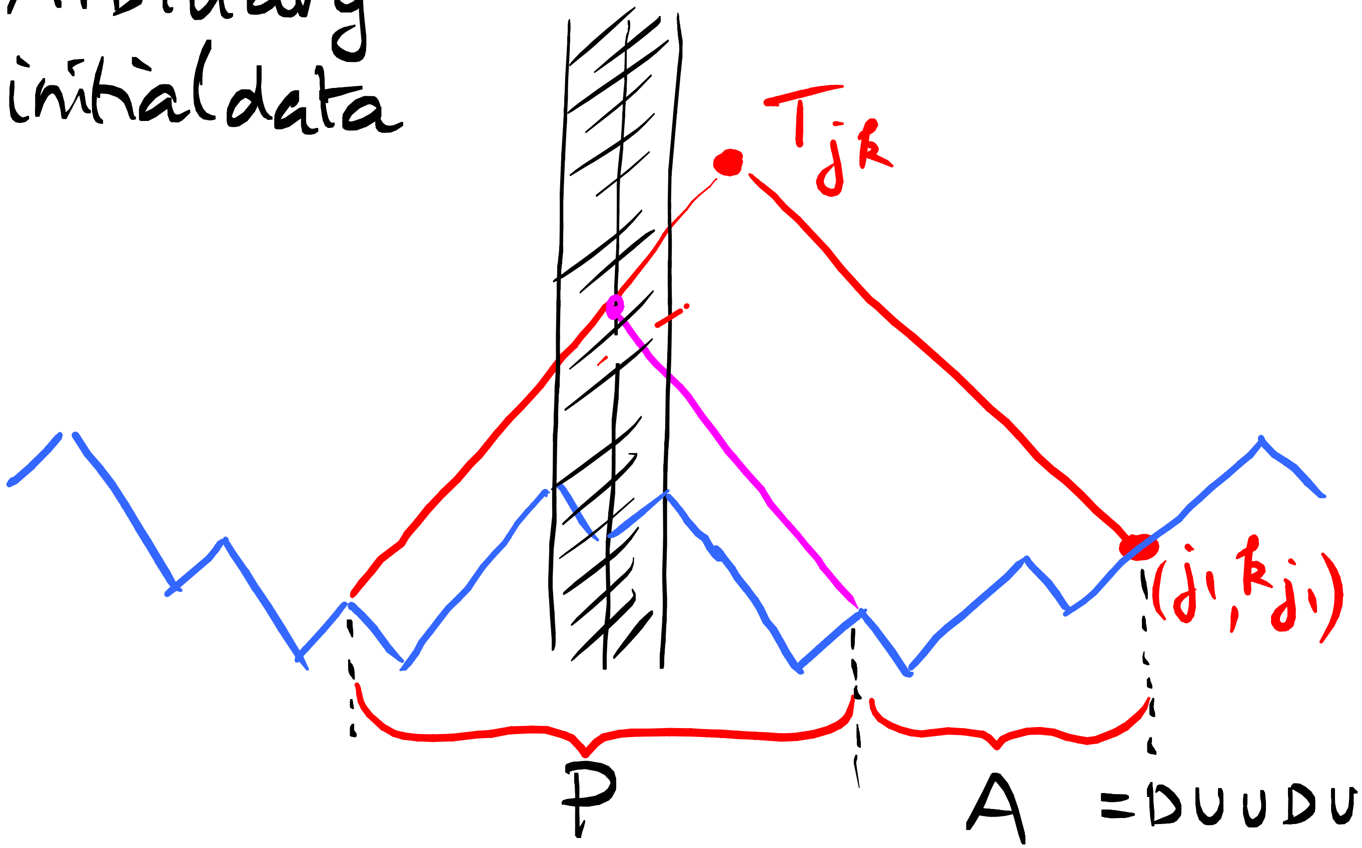


$\boxed{\text{cone is reflected by the wall}}$

$$T_{j,k} = (PA)_{1,1} \cdot T_{j_1, k_{j_1}}$$

POSITIVE

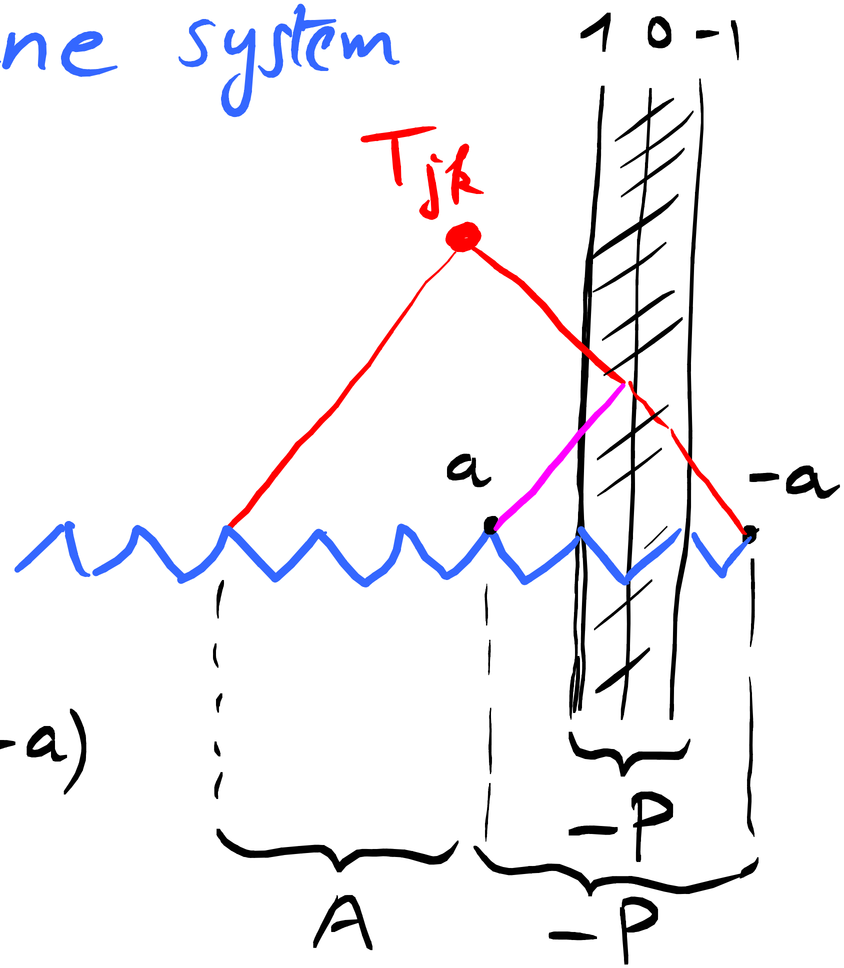
Arbitrary initial data



$$T_{jk} = (PA)_{1,1}\, T_{j',k_{j'}}$$

POSITIVE

The same may be repeated for the left half-plane system



Lemma:

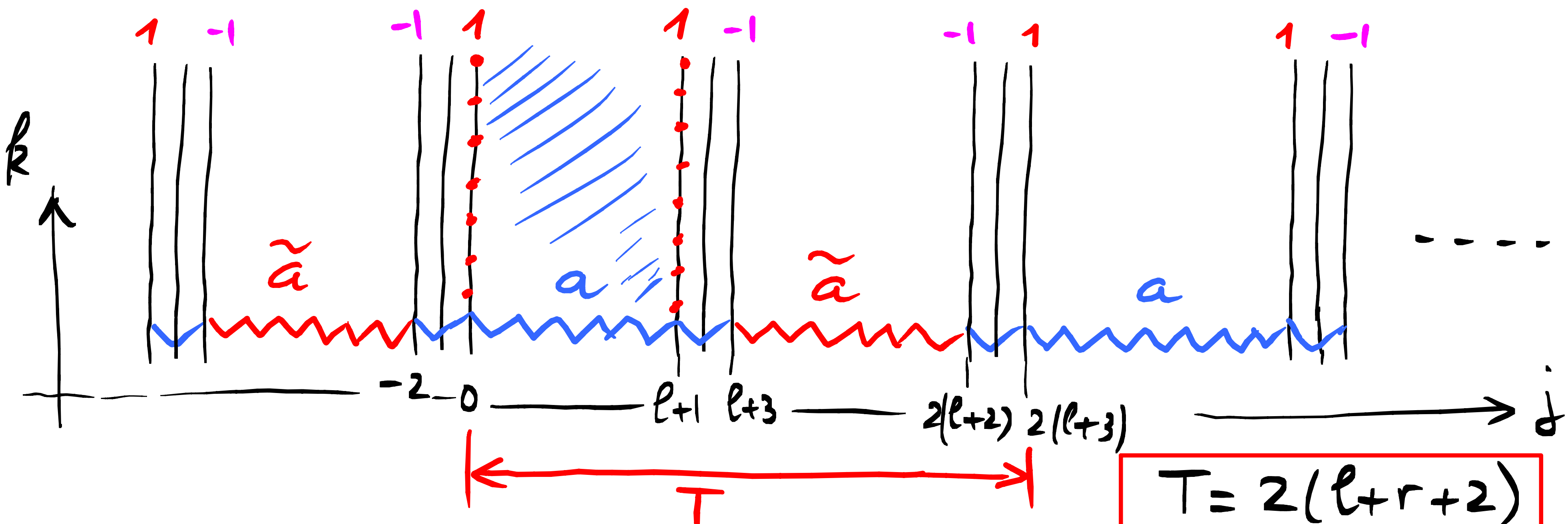
$$D(1,0)\,U(0,-1) = -P$$

$$T_{j,k} = [A(-P)]_{1,1} \times (-a)$$

$$= [AP]_{1,1}\, a$$

POSITIVE

2. Strip: "2 wall" Boundary



$\boxed{T = 2(l+r+2)}$

Remark: the initial data has period T in j.

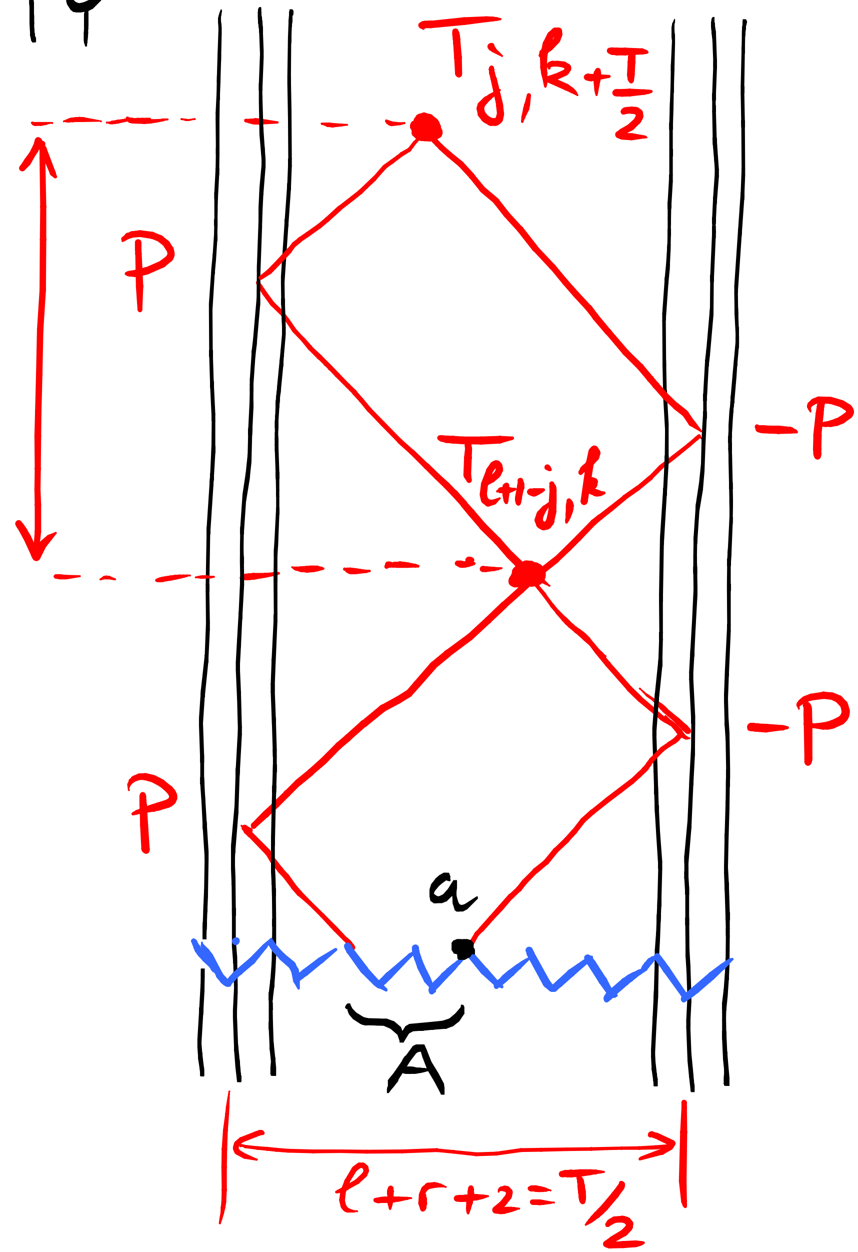
PROOF OF PERIODICITY

"half-period" identity

$T/2 = \ell + r + 2$

Lemma:

$$\boxed{T_{j,k+\frac{T}{2}} = T_{\ell+1-j,k}}$$

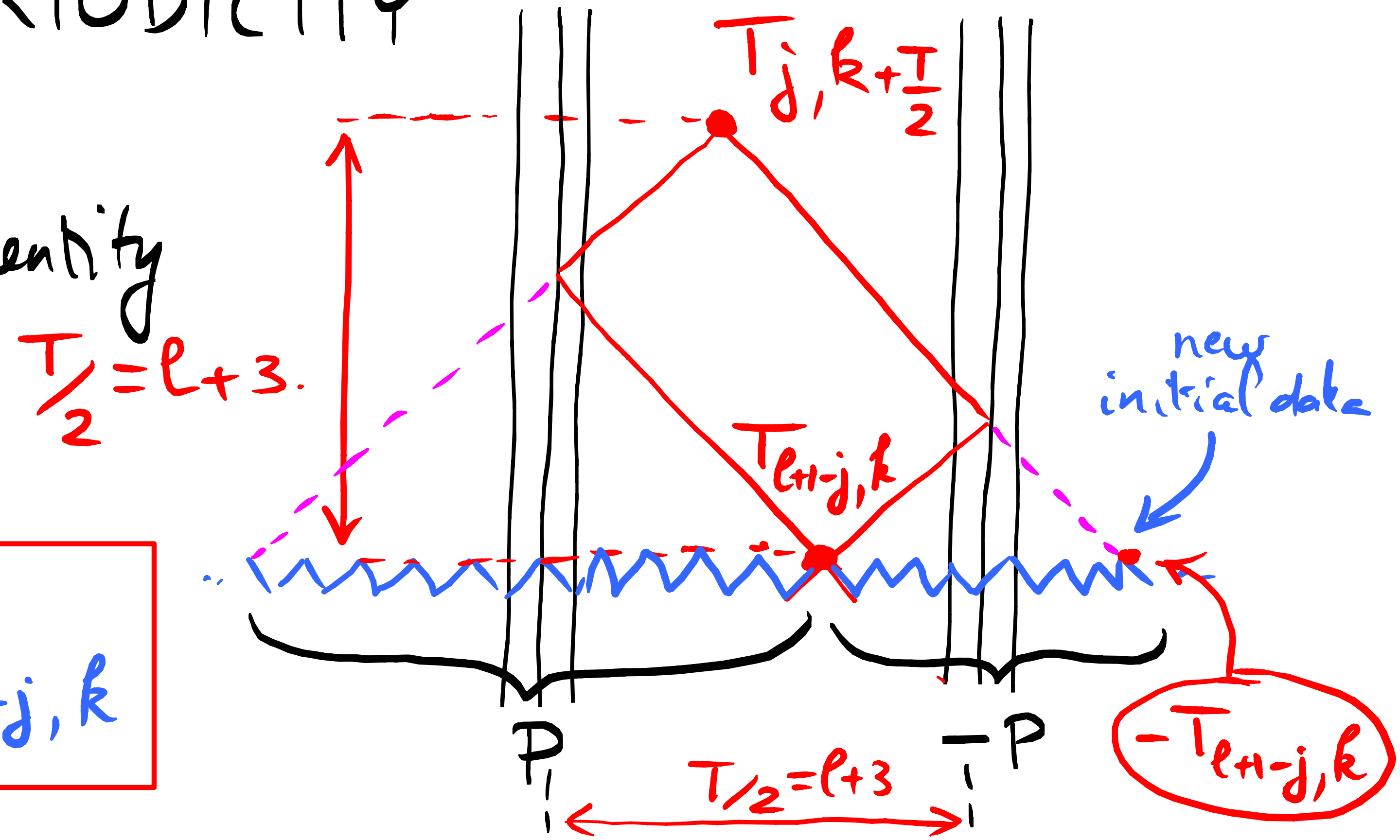


PROOF OF PERIODICITY

"half-period" identity

Lemma:

$$\boxed{T_{j,k+\frac{T}{2}} = T_{\ell+1-j,k}}$$

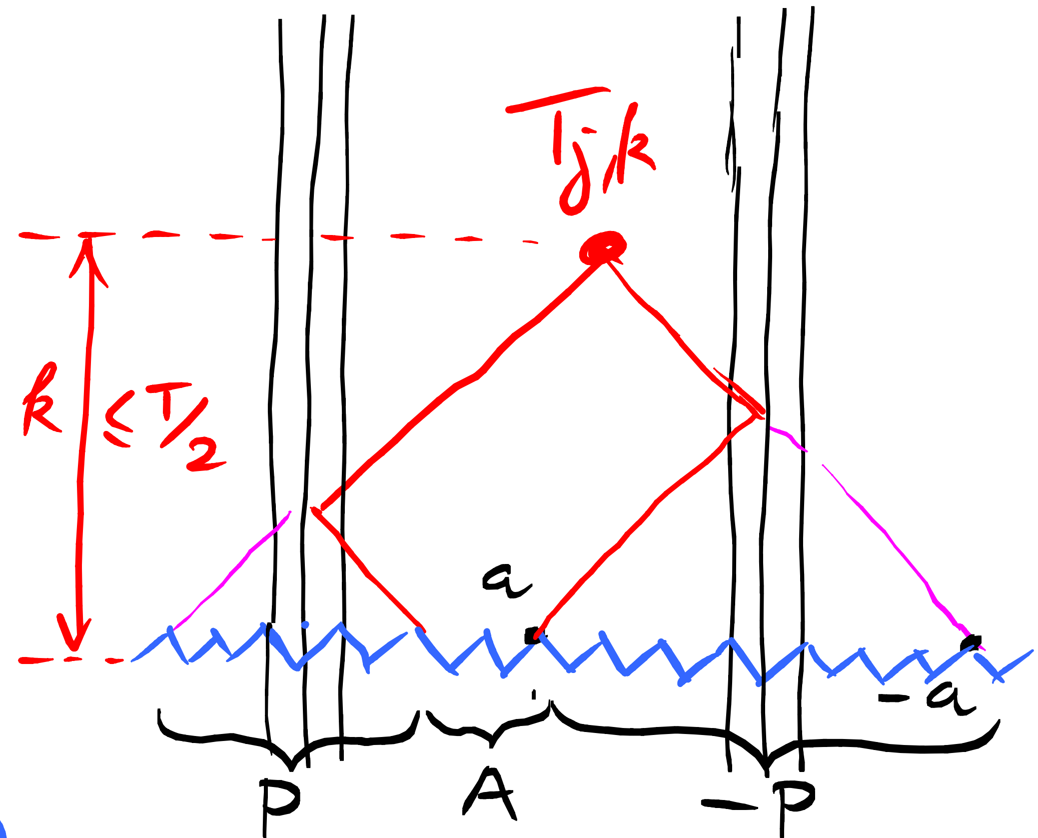


$$T_{j,k+\frac{T}{2}} = \left(P(-P)\right)_{1,1} \times \left(-T_{\ell+1-j,k}\right) = T_{\ell+1-j,k}$$

$\Longrightarrow$ PERIODICITY $\boxed{T = 2(\ell+3)}$

PROOF OF POSITIVITY

use half-periodicity to restrict to $k \leq \frac{T}{2}$



$$T_{j,k} = \left(P\,A\,(-P)\right)_{1,1}(-a)$$

$$= A_{2,2} \cdot a$$

POSITIVE

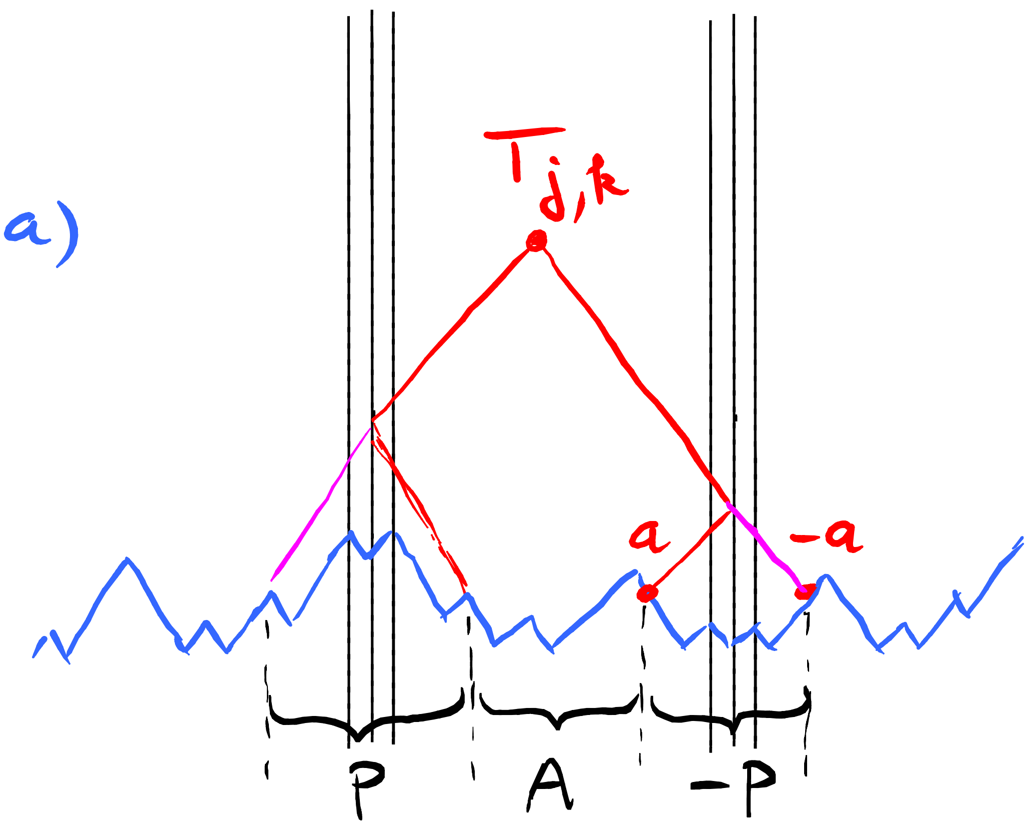
Case of arbitrary initial data zig-zag line:

zig-zag is reflected

$$T_{j,k} = \big(P\,A\,(-P)\big)_{1,1}\,(-a)$$

$$= A_{22} \cdot a$$

$\Rightarrow$ POSITIVE

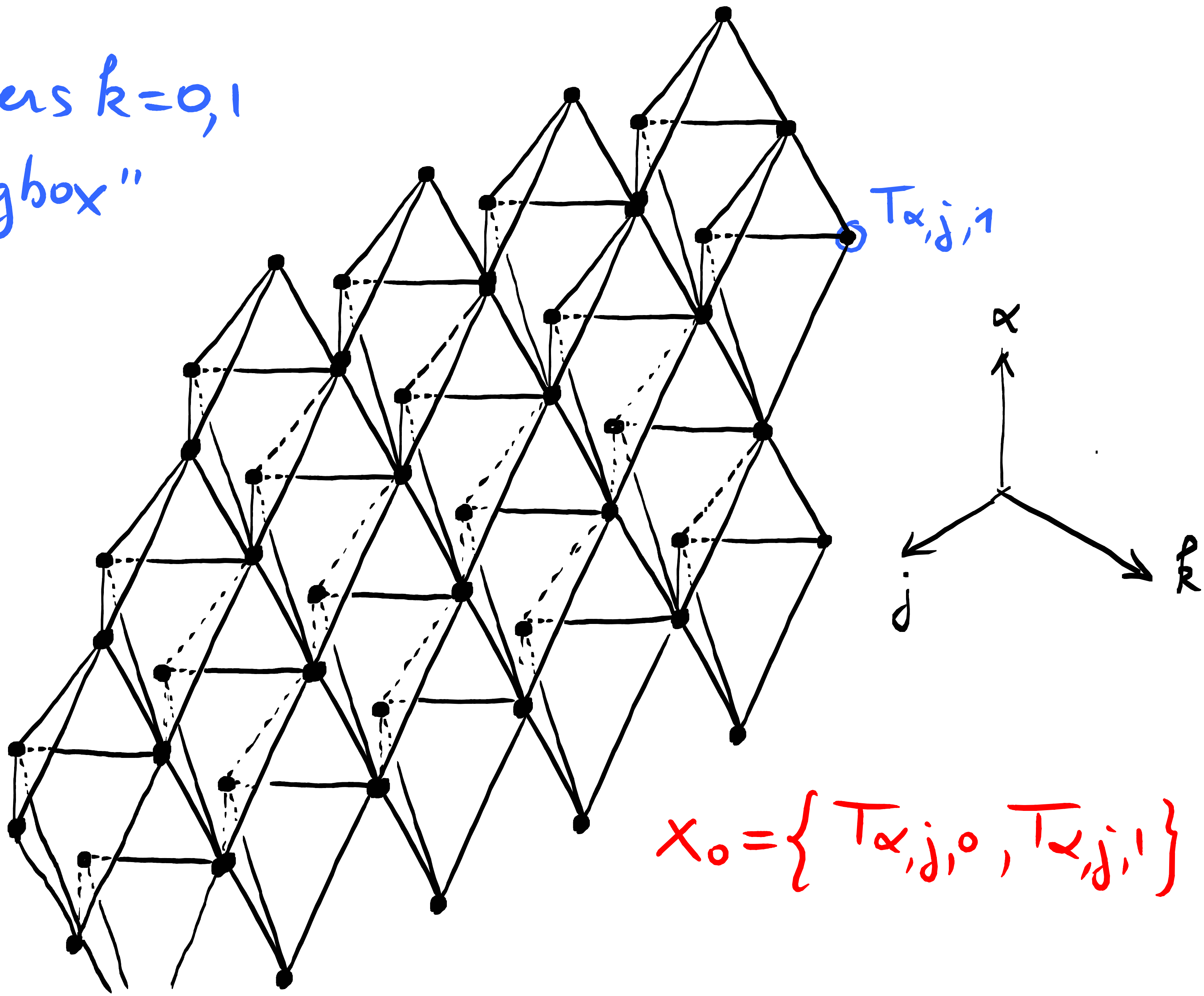


GENERAL CASE: A_r T-system

$$\boxed{\begin{array}{l} T_{\alpha\, j\, k+1}\, T_{\alpha\, j\, k-1} = T_{\alpha\, j+1\, k}\, T_{\alpha\, j-1\, k} + T_{\alpha+1, j, k}\, T_{\alpha-1, j, k} \\ T_{0, j, k} = T_{r+1, j, k} = 1 \qquad 1 \leq \alpha \leq r \quad j, k \in \mathbb{Z} \end{array}}$$

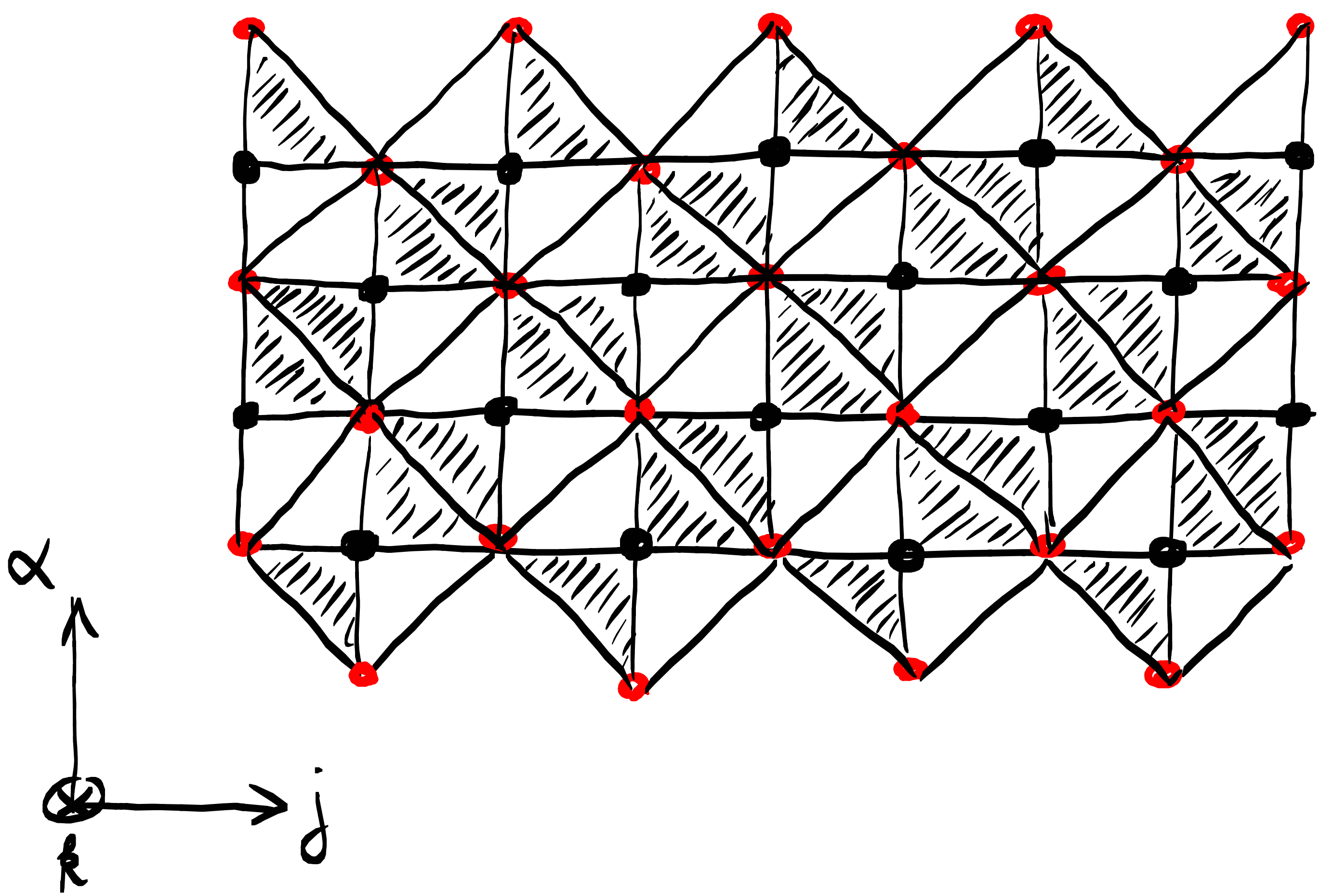
- generalization of D, U matrices
- express solution as a matrix element / network partition function

layers $k=0,1$
of "eggbox"

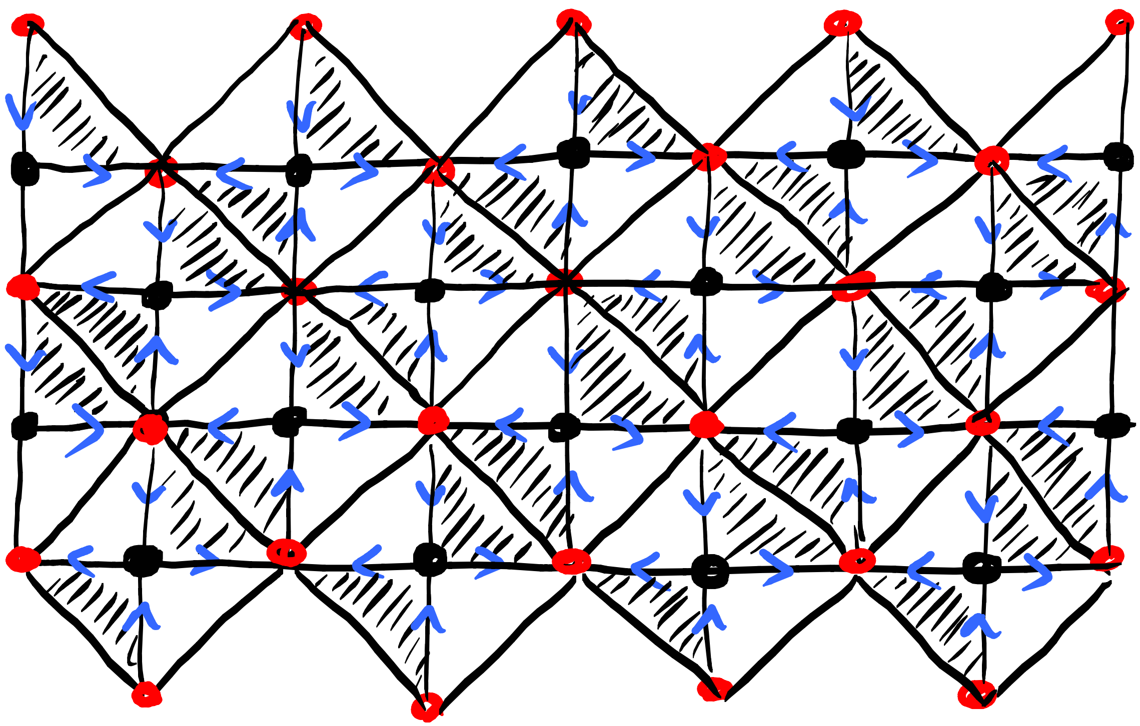


$$X_0 = \{ T_{\alpha,j,0} , T_{\alpha,j,1} \}$$

View from behind: shade triangles as follows:

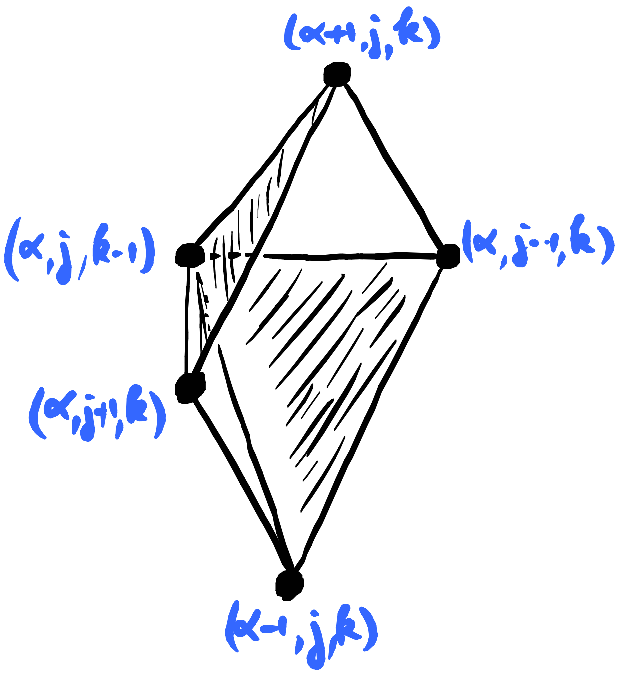


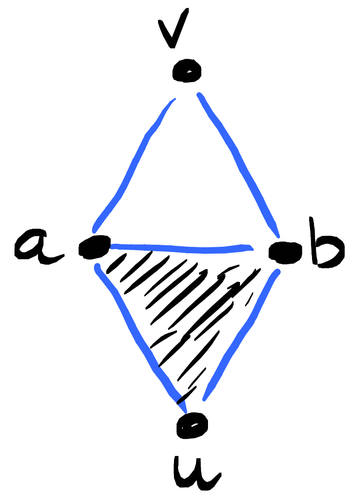
View from behind: shade triangles as follows:



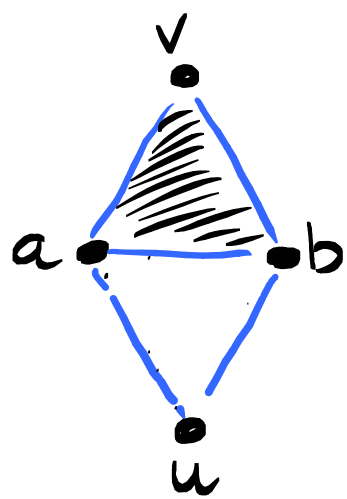
QUIVER = orient all grey faces counter clockwise.

Definition: D & U matrices

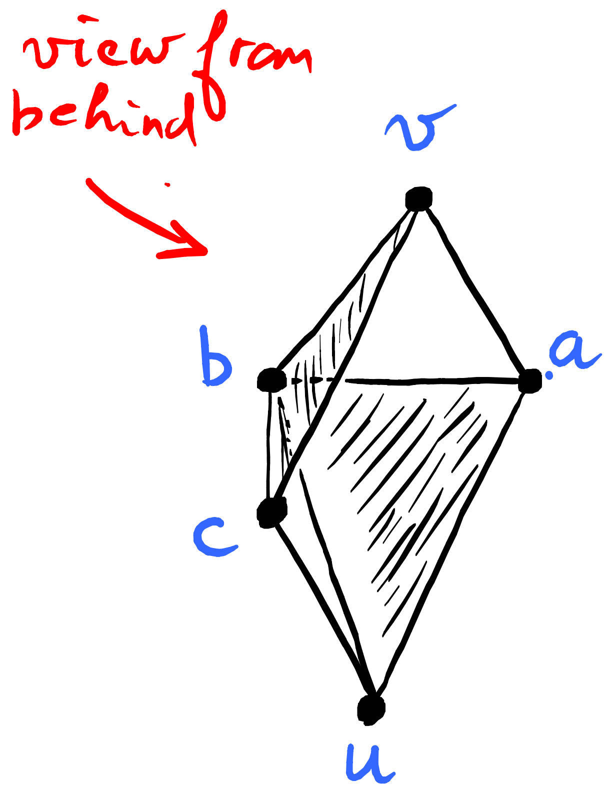




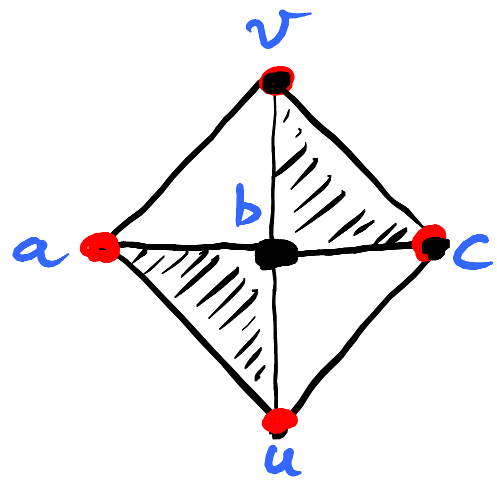
$$\rightarrow \quad D(u,a,b) = \begin{pmatrix} \frac{a}{b} & \frac{u}{b} \\ 0 & 1 \end{pmatrix}$$



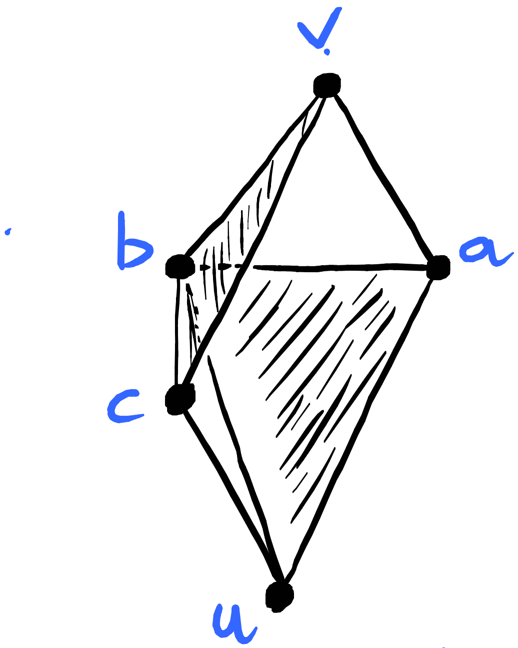
$$\rightarrow \quad U(a,b,v) = \begin{pmatrix} 1 & 0 \\ \frac{v}{b} & \frac{a}{b} \end{pmatrix}$$



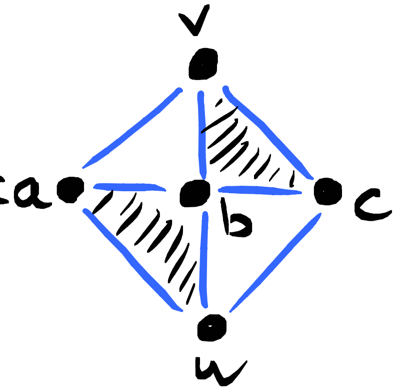
$=$



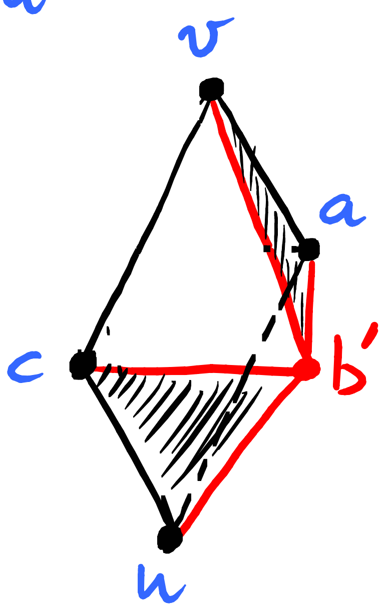
$= D(uab)\bar{U}(bcv)$



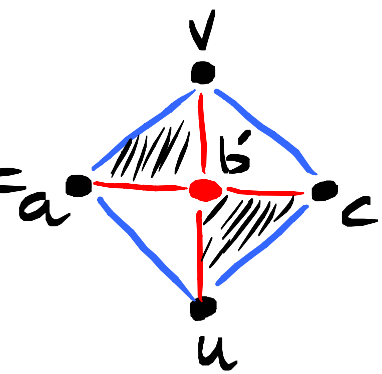
$=$



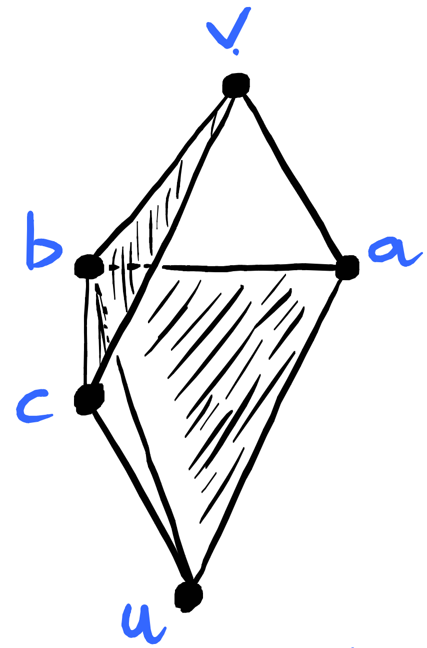
$= D(u,a,b)\,U(b,c,v)$



$=$



$= U(a,b',v)\,D(u,b',c)$



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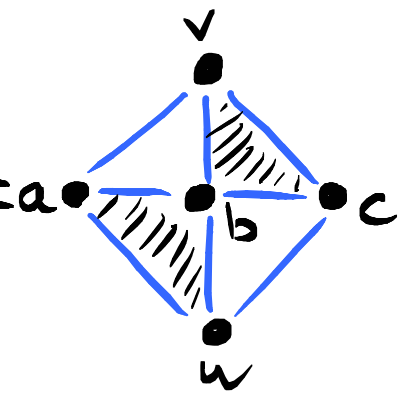
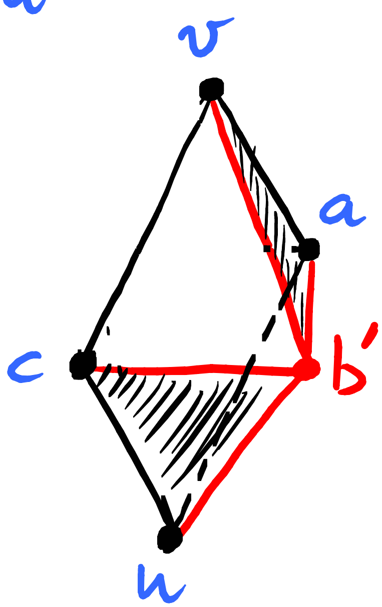


$= D(u,a,b)\, U(b,c,v)$

$\|$

$\Leftrightarrow \boxed{bb' = ac + uv}$



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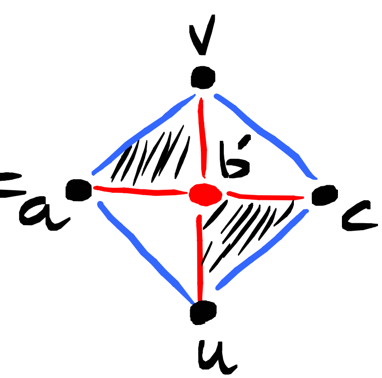


$= U(a,b',v)\, D(u,b',c)$

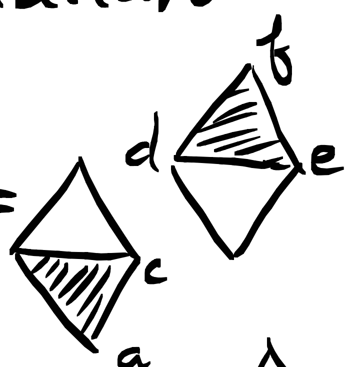
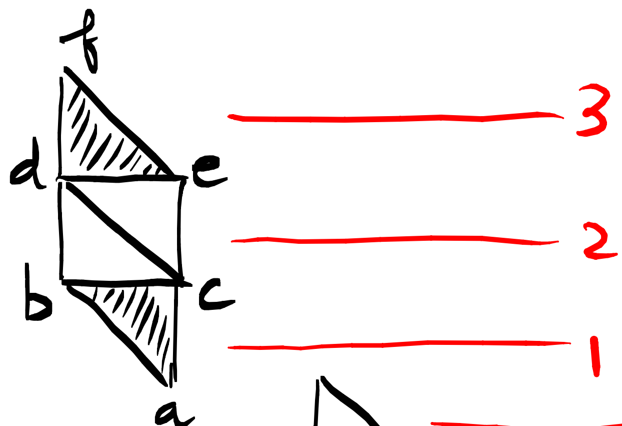
$\Longrightarrow$ REPRESENTATION of T-system

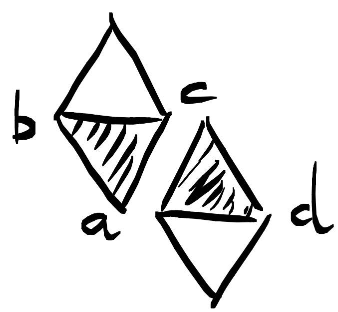
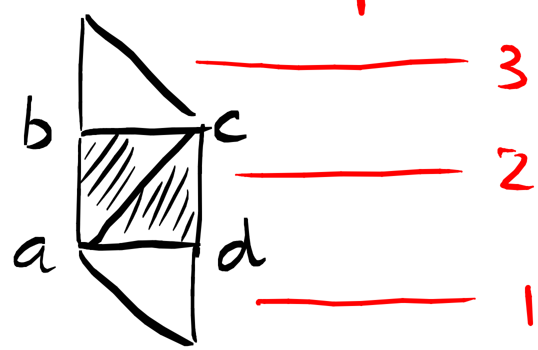
GL_{r+1} EMBEDDING

$$D_{i,i+1}(a,b,c) = \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array}\begin{pmatrix} 1 & & & \\ & \ddots \; 1 & & 0 \\ & & D(abc) & \\ 0 & & & 1 \ddots 1 \end{pmatrix}$$

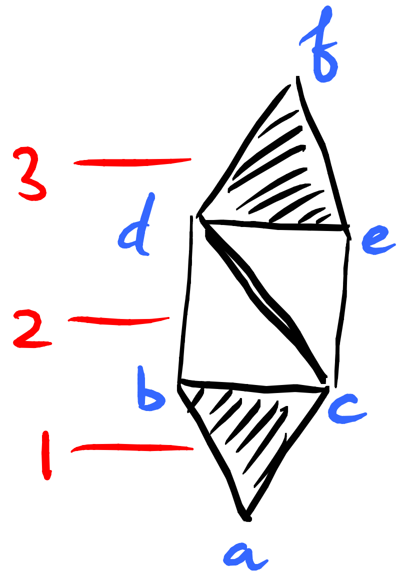
$$U_{i,i+1}(a,b,c) = \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array}\begin{pmatrix} 1 & & & \\ & \ddots \; 1 & & 0 \\ & & U(abc) & \\ 0 & & & 1 \ddots 1 \end{pmatrix}$$

Product representation

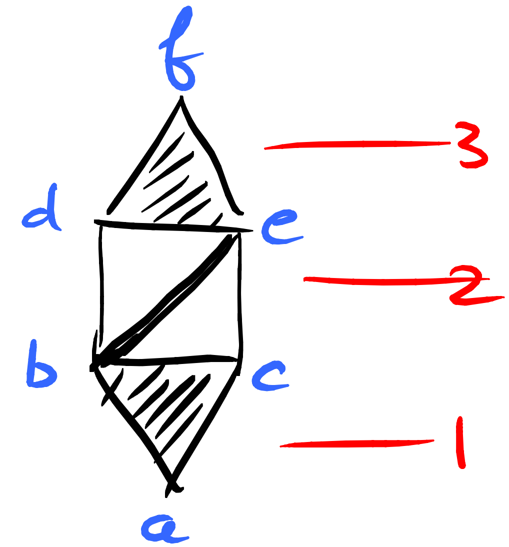
$D_{12}(abc)\,U_{23}(def) =$  $\equiv$ 

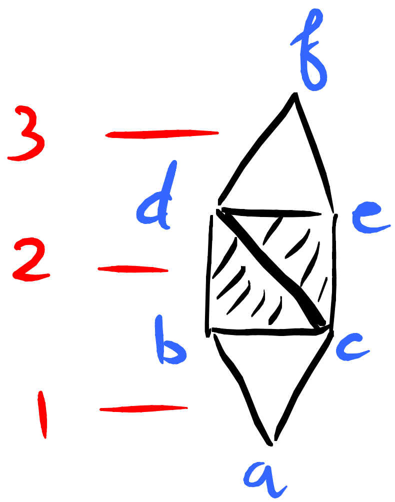
$D_{23}(abc)\,U_{12}(adc) =$  $\equiv$ 

OTHER PROPERTIES OF D, U MATRICES

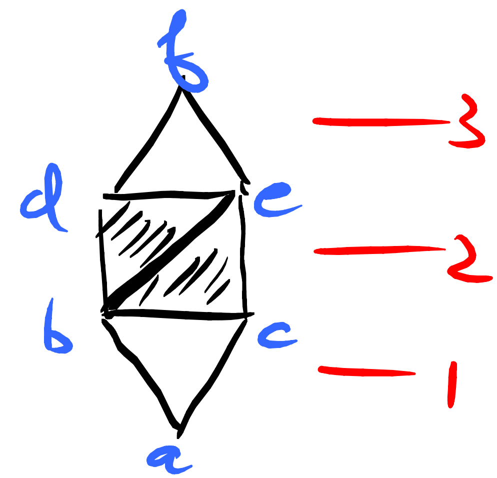


$$D_{12}(abc)\,U_{23}(def) = U_{23}(def)\,D_{12}(abc)$$





$$U_{12}(bcd)\,D_{23}(cde) = D_{23}(bde)\,U_{12}(bce)$$



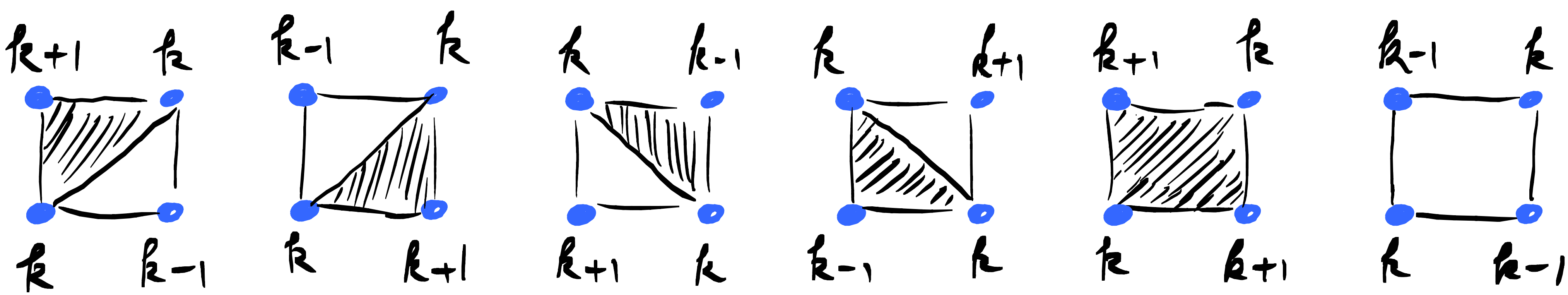
in colored squares diagonal is arbitrary

D, U MATRIX PRODUCT ASSOCIATED TO ANY STEPPED SURFACE

k = height variable

$$\begin{cases} |k_{\alpha,j+1} - k_{\alpha,j}| = 1 \\ |k_{\alpha+1,j} - k_{\alpha,j}| = 1 \end{cases}$$

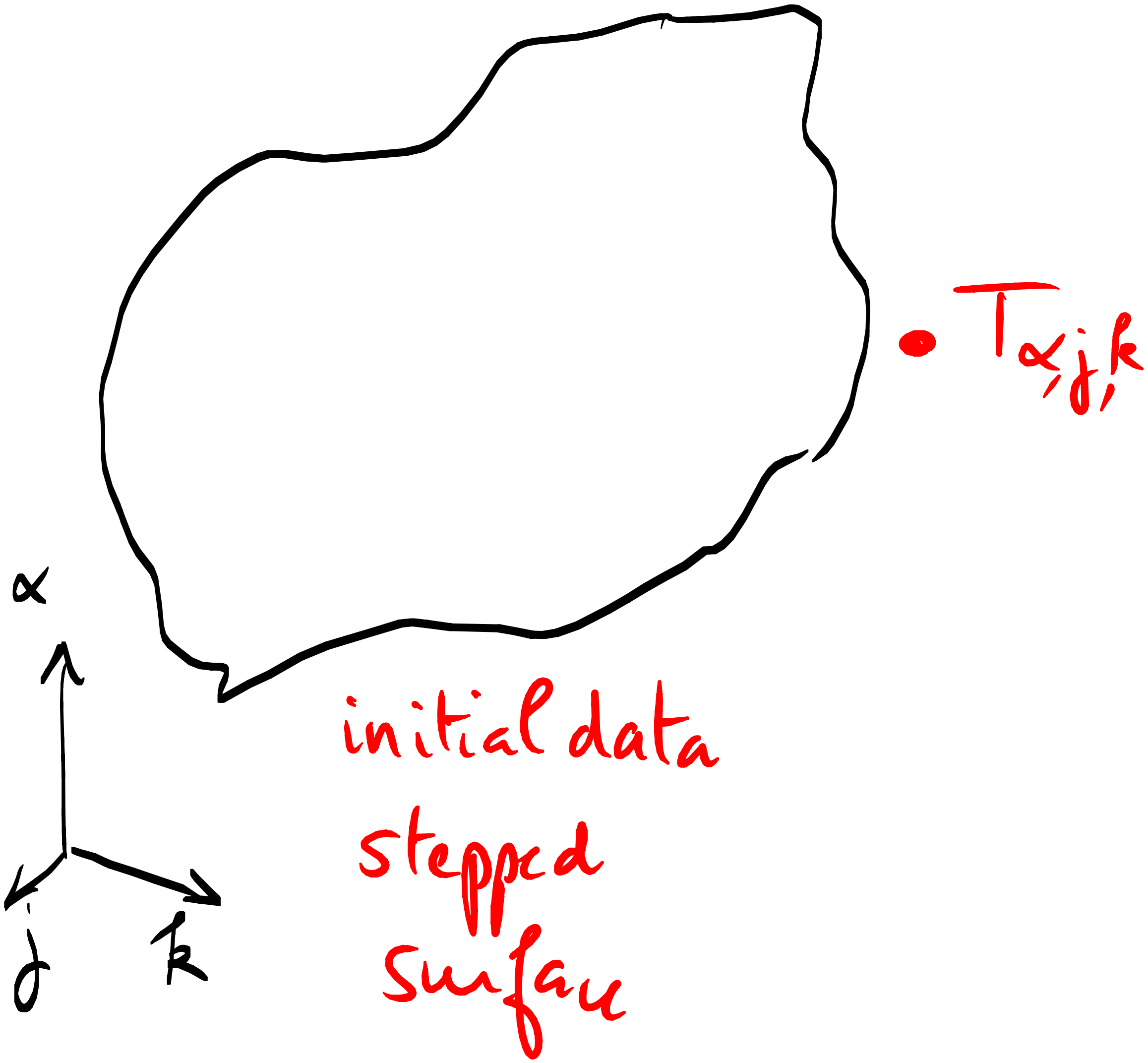
$\Rightarrow$ 6 local configurations for each square in (αj) plane



$\boxed{\text{diagonal joins vertices of equal height}}$

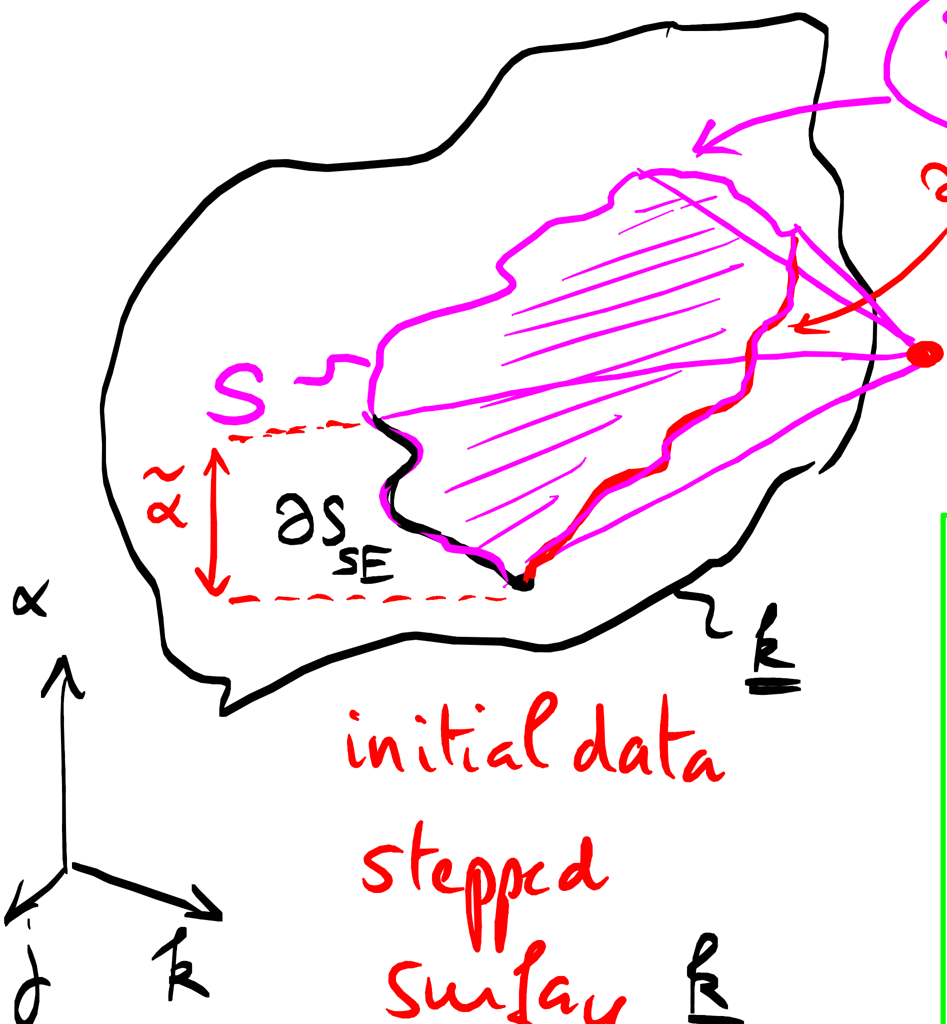
2 choices of diagonal

SOLUTION



SOLUTION

S = "pyramidal" projection of $(\alpha j k)$ onto $\underline{\underline{k}}$



initial data stepped surface $\underline{\underline{k}}$

$T_{\alpha j k}$

$$\left(S = \underbrace{\Pi_{\alpha j k}}_{\{x y z \,\mid\, |x-\alpha| + |y-j| \le |z-k|\}} \cap \underline{\underline{k}} \right)$$

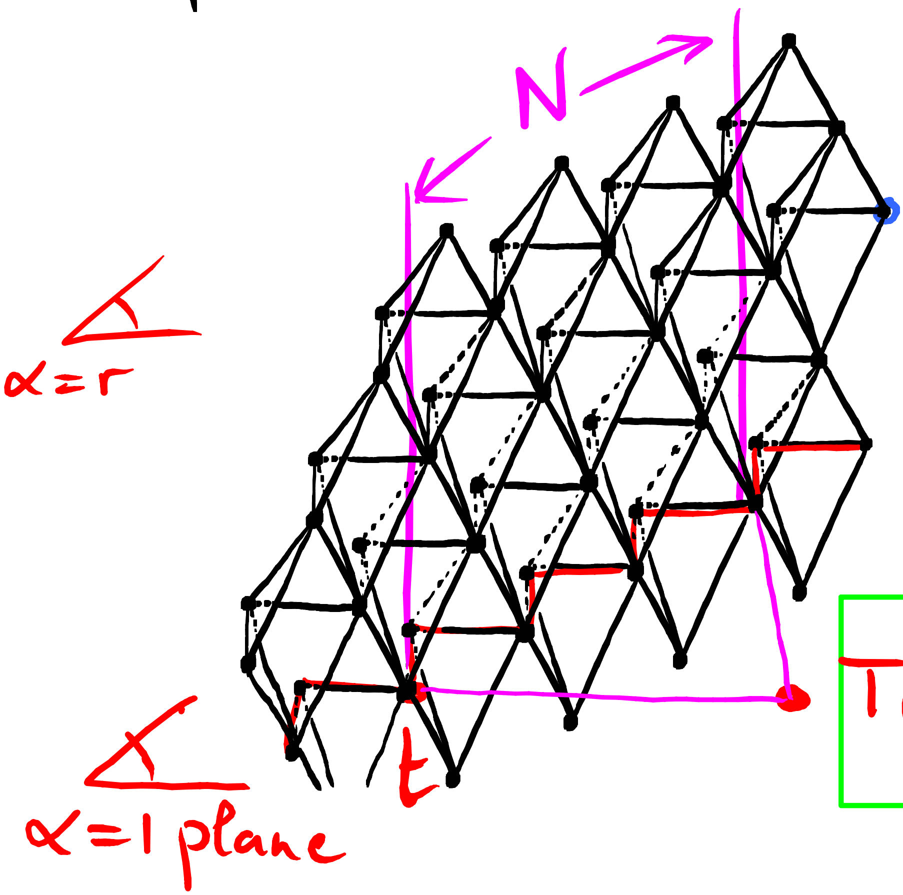
Thm: $T_{\alpha j k}$ = principal $\tilde{\alpha} \times \tilde{\alpha}$ minor of the product of D's and U's corresponding to $S \times \prod_{\partial S_{SW}} t^{-1} \times \prod_{\partial S_{E}} t$

initial values on SW & SE border of S

A_r - T-SYSTEM

POSITIVITY: follows from the formula for $\alpha=1$ and det identity.

Special case $\alpha=1$: formula is analogous to A_1 case

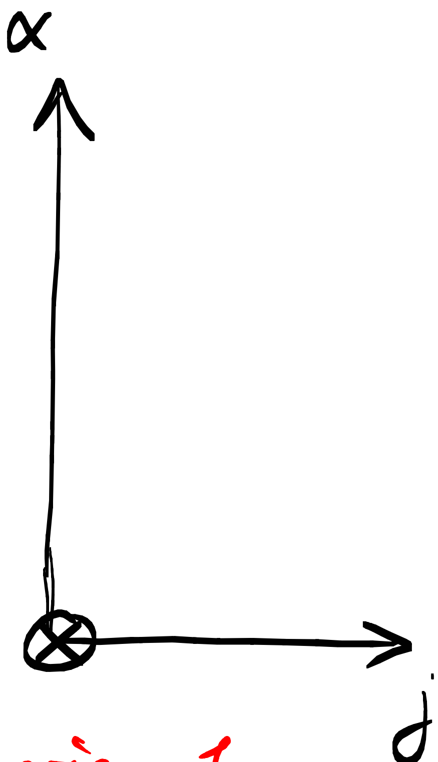


Product of D's and U's

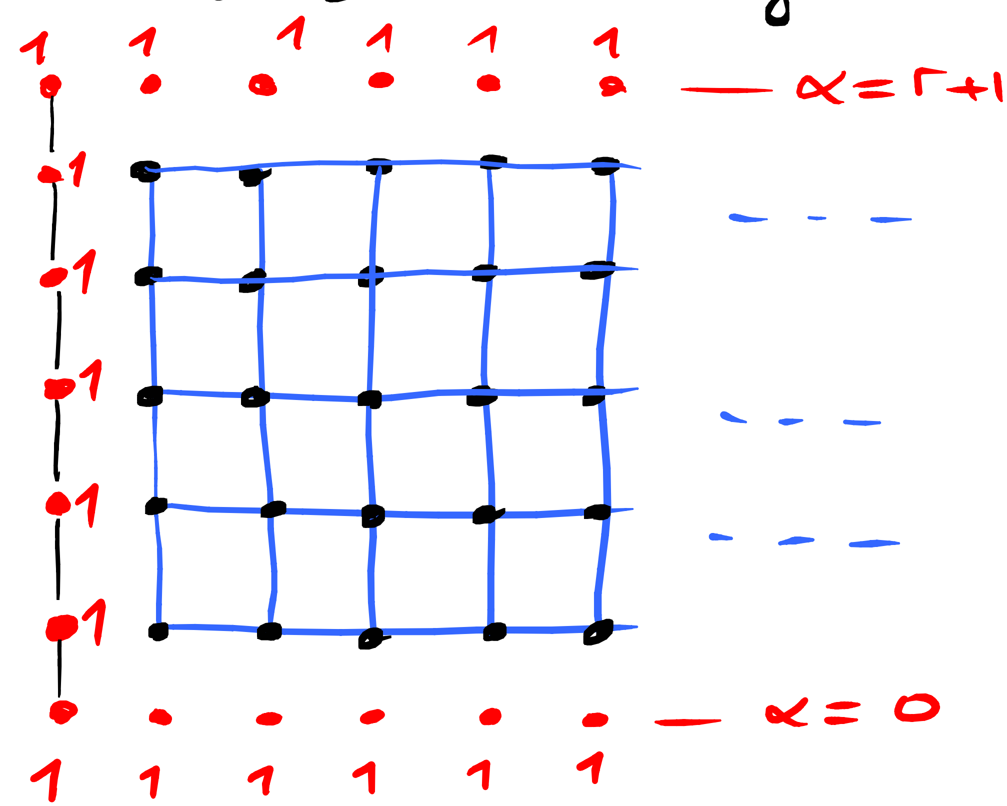
$$\boxed{T_{1,j,k} = [N]_{1,1} \times t}$$

PERIODICITY (flat stepped surface)

One wall boundary condition for T-system:



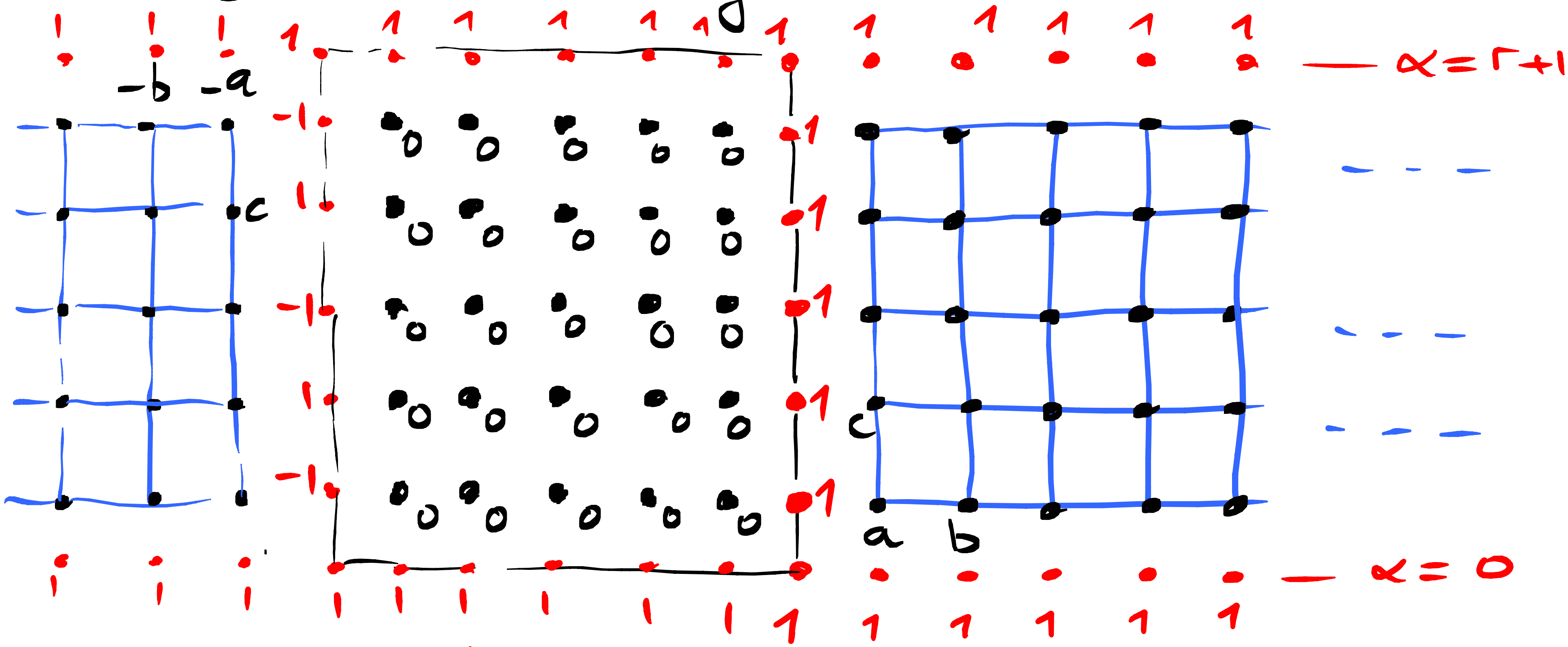
view from behind $\underline{h}$



The solution in the right $\frac{1}{2}$-space is same as that in full space with

PERIODICITY (flat stepped surface)

One wall boundary condition



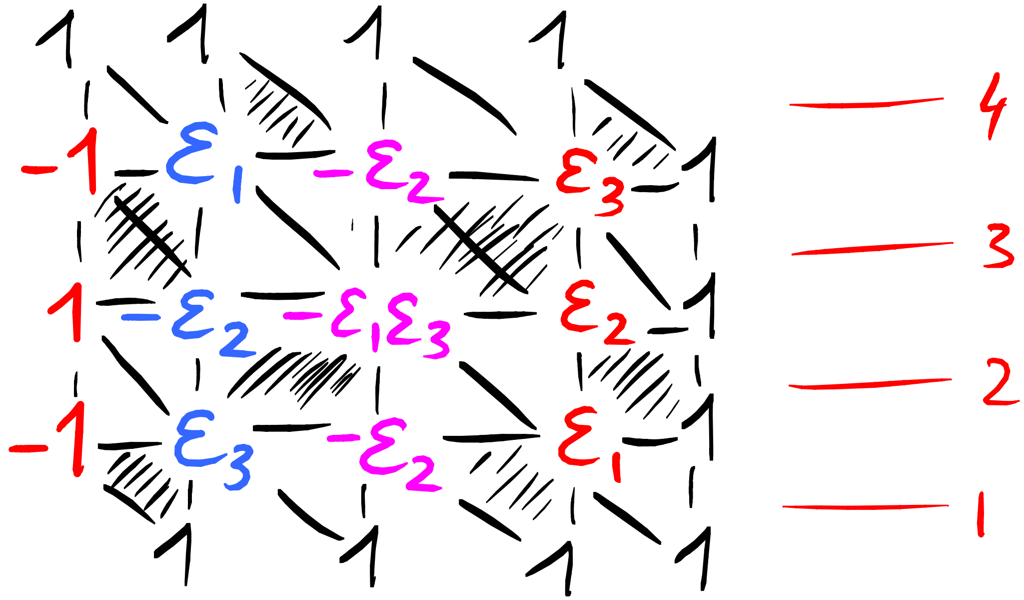
suitably reflected initial data

$$\boxed{t_{r+1-j,\,r-1-k} = (-1)^{rj}\, t_{jk}}$$

$(r=5 \text{ here})$

Ex: $r=3$

$P(\{\varepsilon\}) =$



(Product of D, U matrices)

$$\boxed{\lim_{\varepsilon_1\varepsilon_2\varepsilon_3\to 0} P(\{\varepsilon\}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = P}$$

More generally $P_{ij} = \delta_{i+j,r+2}(-1)^{(r-1)(i-1)}$

$$\boxed{\text{Lemma: } V_{r+1-i,r+2-i}((-1)^i c, b, a)\, P\, U_{i,i+1}(a,b,c) = P}$$

- This allows to repeat the A_1 case for $T_{1,j,k}$, as: $\boxed{g(t)}\boxed{P}\boxed{t} = \boxed{P}$

$\Rightarrow$ POSITIVITY (1 wall)

- Box boundary (2 walls)

$$\cdots\ \boxed{\tilde{a}}\boxed{P}\underbrace{\boxed{a}}_{\ell+1}\underbrace{\boxed{(-1)^r P}}_{r+1}\boxed{\tilde{a}}\boxed{P}\ \cdots$$

the A_r T-system with these initial data has the same solution as that in the Box

- This allows to repeat the A_1 case for $T_{1,j,k}$, as: $\boxed{g(t)}\boxed{P}\boxed{t} = \boxed{P}$

$\Rightarrow$ POSITIVITY (1 wall)

- Box boundary (2 walls)

$$\cdots \boxed{g(a)}\underbrace{\boxed{P}\underbrace{\boxed{a}}_{\ell+1}\underbrace{\boxed{(-1)^r P}}_{r+1}\boxed{g(a)}}_{T=2(\ell+r+2)}\boxed{P}\cdots$$

initial data is T-periodic.

- Same mechanism for $T_{1,j,k}$ as in A_1 case

$\Rightarrow$ POSITIVITY

PERIODICITY $\boxed{T = 2(\ell + r + 2)}$

- then $T_{\alpha j k} = \alpha \times \alpha$ determinant of the T_1's,

$\Rightarrow$ also periodic.

- via LGV, $T_{\alpha j k} = \alpha$ non-intersecting paths on network (possibly reduced by reflections)

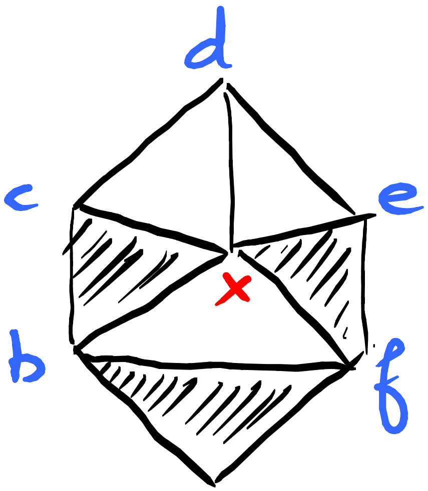
$\Rightarrow$ also positive.

CONCLUSION

- everything reduces to D, U matrices and the mutation relation $\boxed{DU = U'D'}$
- link to Yang-Baxter eqn for networks (integrability).
- link to Dimer models / Domino Tilings

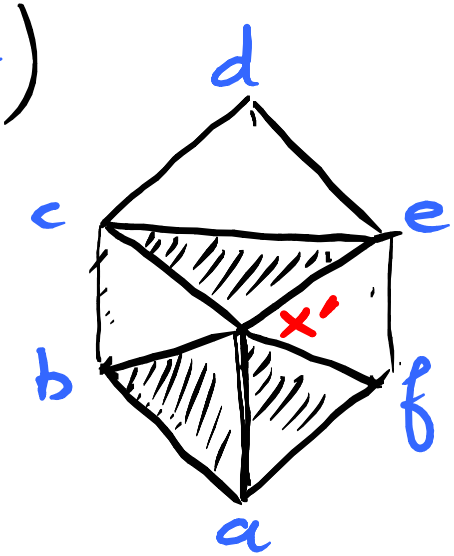
  (D, U → transfer matrices for the Tiling)
- other groups / coefficients as well [DF]
- non-commutative extensions q A, T [DF-K]

EX 1: Generalized Yang-Baxter eqn for D matrix



$$D_{23}(bcx)\,D_{12}(abf)\,D_{23}(fxe)$$

$$= D_{12}(abx)\,D_{23}(x'ce)\,D_{12}(ax'f)$$

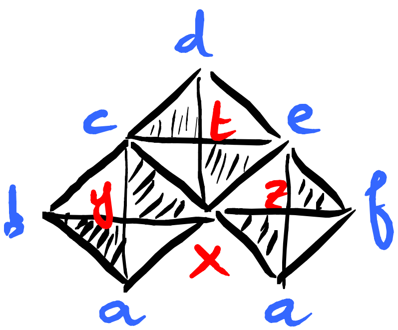


$$\Longleftrightarrow \qquad \boxed{xx' = be + cf}$$

Allows to have a "cube melting" octahedron recurrence

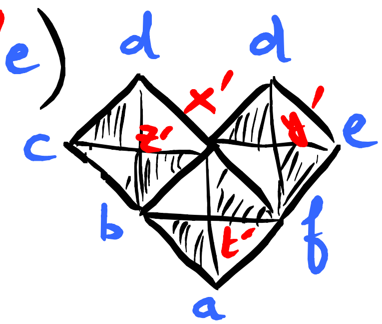
(cf R. Kenyon's talk)

EX2. D,U matrix representation of ~~hexahedron~~ garnet eq. via gen DU Yang-Baxter eqn:



$$D_{12}(aby)\,U_{12}(yxc)\,U_{23}(ctd)\,D_{23}(xte)\,D_{12}(axz)\,U_{12}(zfe)$$

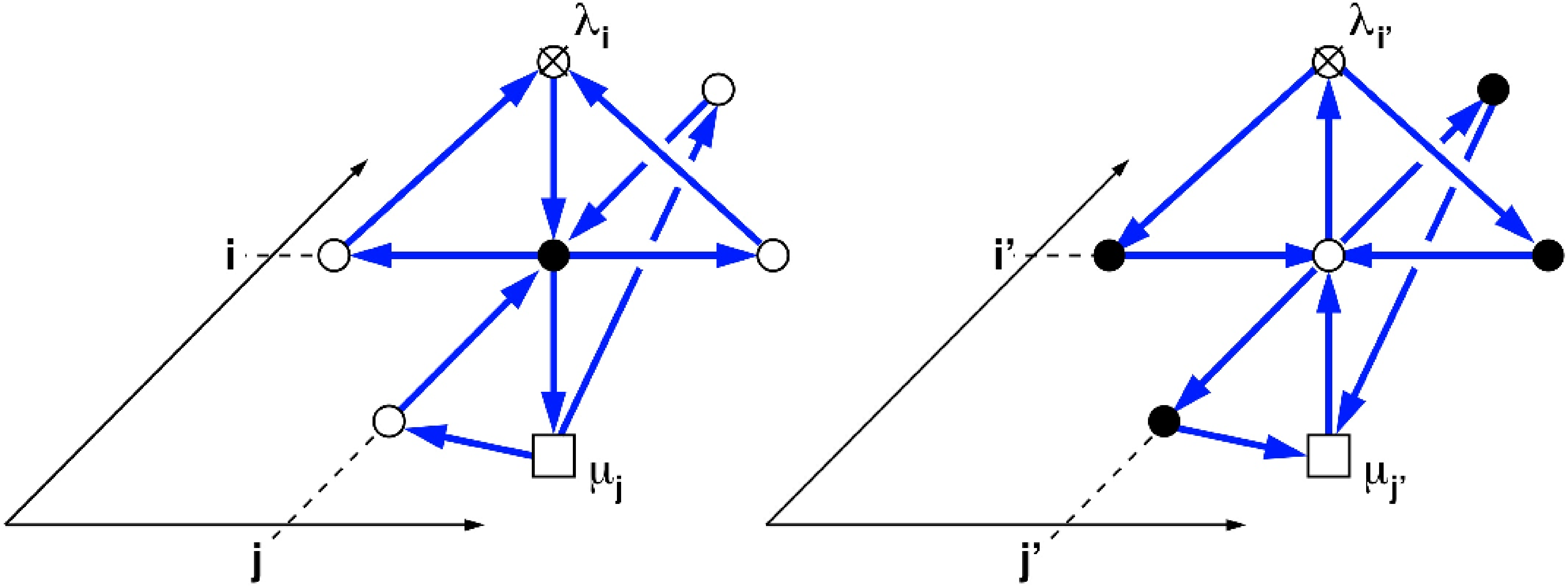
$$= U_{23}(cz'd)\,D_{23}(bz'x')\,D_{12}(abt')\,U_{12}(t'fx')\,U_{23}(x'y'd)\,D_{23}(fy'e)$$



$\Longleftrightarrow$

$$\boxed{\begin{aligned} xtt' &= xad+ace+yzt \\ xyy' &= xeb+ace+yzt \\ xzz' &= xfc+ace+yzt \\ x^2yztx'^2 &= (xad+ace+yzt)(xeb+ace+yzt) \\ &\quad + xf\big((xb+ae)(xd+ae)+yztc\big) \end{aligned}}$$

T-system with coefficients



$$\boxed{T_{i,j,k+1}T_{i,j,k-1} = \lambda_i T_{i+1,j,k}T_{i-1,j,k} + \mu_j T_{i,j+1,k}T_{i,j-1,k}}$$

Allows to define the $\vec{\lambda},\vec{\mu}$ – determinant of a $n \times n$ matrix A by picking initial data

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & a_{1n} & 1 & & \\
 & 1 & & 1 & & 1 & \\
1 & a_{11} & 1 & \cdots & 1 & a_{nn} & 1 \\
 & 1 & & 1 & & 1 & \\
 & & 1 & a_{n1} & 1 & & \\
 & & & 1 & & &
\end{array}$$

$$\boxed{|A|_{\vec{\lambda}\vec{\mu}} = T_{0,0,2n-1}}$$

$\downarrow$

explicit expression in terms of ASM – 6vertex configs.