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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SRHERMES@BRANDEIS.EDU

Speaker's Name: A. Felikson

Talk Title: Cluster Algebras from Orbifolds

Date: 10 / 31 / 12 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Cluster Algebras, Triangulations, Orbifolds, Hyperbolic Geometry, Surface Cluster Algebras, Skew-symmetrisability

Please summarize the lecture in 5 or fewer sentances: The speaker introduced a generalization of surface cluster algebras to surfaces with cone points. This allows the construction of surface cluster algebras to be extended to skew-symmetrisable matrices. She gave a complete local description of these algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Cluster Algebras from Orbifolds

A. Felikson November 1, 2012

Joint with Shapiro, Tumarkin.

We want to generalize the construction of Fomin, Shapiro, Thurston [FST], [FT]. Parts of the construction are sketched by Chekhov and Mazzocco.

1 Motivation

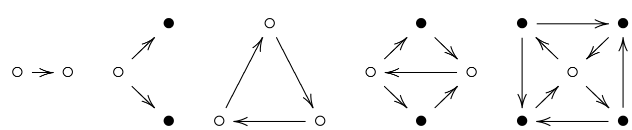
Recall the input data for a cluster algebra is a skew-symmetriseable integer matrix B; i.e. there exists a diagonal matrix D with integer entries so that DB is skew-symmetric. Typically proofs about cluster algebras are given by proving the result for symmetric matrices and then extending to skew-symmetriseable matrices.

Theorem 1.1. A cluster algebra with skew-symmetric exchange matrix B is of finite mutation type if either:

1. B is rank 2,
2. B arises from a triangulated surface,
3. or B is one of 11 exceptional types.

This second condition is what we are going to generalize based on the following:

1. skew symmetric matrices are in one-to-one correspondence with quivers
2. a quiver corresponds to a triangulated surface if and only if it is block decomposable, i.e. it can be obtained from

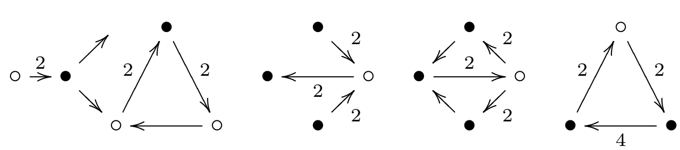


(where we can join two $\circ$'s but cannot join $\bullet$'s to anything).

More generally we have: Skew-symmetriseable matrices can be represented by a diagram (not a bijection). The diagram has n vertices for B $n \times n$ and there is an arrow $i \to j$ of weight $-b_{ij}b_{ji}$ if and only if $b_{ji} > 0$.

Theorem 1.2. A diagram is of finite mutation type if it is either:

1. order 2,
2. one or $11 + 7$ exceptional types,
3. or block decomposable as above with additional blocks



These new diagrams are obtained by "collapsing" the old diagrams to get an orbifold.

2 The Construction

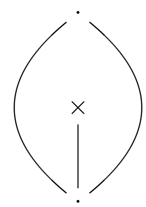
2.1 Triangulated Orbifolds

Definition 2.1. For us, an *orbifold* is a surface $\mathcal{O}$ of genus g with marked points, and boundary components (each of which contains at least one marked point) together with some *orbifold points*—that is, cone points with angle π (i.e. an order 2 orbifold point). We think of it as some special marked point (this will affect the triangulations).

Definition 2.2. A *triangulation* of $\mathcal{O}$ is a maximal set of non-intersecting arcs where:

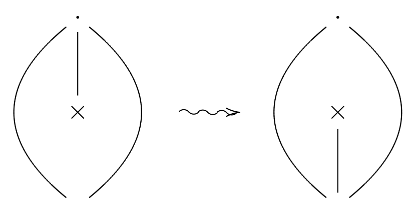
1. ends of arcs are in marked points or orbifold points,
2. at most one edge goes into any given orbifold point,
3. usual rules for tagging, but no tags at the ends of arcs at orbifold points.

Example 2.3.



is a triangle with one orbifold point (denoted $\times$ above); both "sides" of the vertical line are the same side of a single arc.

We need a notion of flipping edges in triangulated orbifolds. We have the usual flips and a new move for oribifold points:



Proposition 2.4. Flips act transitively on triangulations of $\mathcal{O}$.

2.2 From Triangulations to Diagrams

For ordinary triangles, the procedure is the same as for ordinary surfaces (i.e. put vertices on the edges and arrows along the orientation of the triangle). For an orbifold point, we get arrows of weight 2 going to and from the edge $\cdot \!\!-\!\!\!-\!\!\!-\!\! \times$. With this diagram, flips correspond to diagram mutation.

Diagrams from orbifolds are exactly the diagrams which are block decomposable into the blocks corresponding to skew-symmetriseable matrices.

2.3 Weighted Orbifolds

To get a bijection between matrices and diagrams, we need to *weight* the vertices of the diagrams (i.e. label with numbers d_i).

In connected block decomposable diagrams, one can show:

1. For $\circ$ vertices, $d_i = 2$ for fixed w.

2. For $\bullet$ vertices $d_i = w, 2w$, or $w/2$ for fixed w.

Only the $\bullet$ vertices correspond to the arcs going into orbifold points. Thus we need to label the orbifold points by 2 or 1/2.

2.4 Hyperbolic Orbifolds with all Orbifold Points of Weight $1/2$

The orbifolds above can be given a hyperbolic metric so that arc lengths satisfy the usual cluster algebra exchange relation:

Take ideal hyperbolic triangles for the triangles in the triangulation. For orbifold points, take isosceles hyperbolic triangles and fold the one edge in half. Decorate cusps with horocycles. Define lengths as usual; i.e. the length $\ell(\gamma)$ of an arc γ is the length of the finite segment between the horocycles.

Define $\lambda = e^{\ell/2}$. The cluster variables $\lambda(\gamma)$ satisfy the usual cluster algebra exchange relation.

For weight 2 points, replace orbifold point with a special marked point (same triangulation) with the self-conjugate horocycle of length 1.

2.5 Unfolding

We want to have some (usual) surface $p : \Sigma \to \mathcal{O}$ mapping to our orbifold so that the orbifold triangulation comes from a triangulation upstairs and flips downstairs come from flipping the covering edges upstairs. We also want λ-lengths to match up. Such a surface with is called an *unfolding* of $\mathcal{O}$.

Theorem 2.5. This surface unfolding exists unless $\mathcal{O} = S^2$ and contains exactly one orbifold point of weight 1/2.

3 Results

1. Growth rates calculated (unfold and use [FST])

2. Positivity (Musiker, Schiffler, Williams)

3. Sign coherence of c-vectors (Derksen, Weyman, Zelevinsky)

4. Bases [MSW] (for orbifolds without punctures and at least 2 boundary marked points)

5. Positivity of d-vectors [FST] (can only prove for initial triangulations without a loop bounding an orbifold point of weight 1/2.)