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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SRHERMES @ BRANDEIS. EDU

Speaker's Name: Lauren Williams

Talk Title: Combinatorics of KP Solitons from the Real Grassmannian

Date: 10 / 31 / 12 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: KP-equation; solitons, surface waves, tropical geometry, Grassmannians, Combinatorics, Graphs

Please summarize the lecture in 5 or fewer sentances: The speaker introduced several solutions to the KP equation parametrised by Plücker coordinates on the Real Grassmannian. She showed that the contour plot of a given solution can be encoded in a decorated planar graph, and the asymptotic behavior of a solution is classified by various combinatorial stratifications of the Grassmannian.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Combinatorics of KP solitons from the real Grassmannian

Lauren K. Williams, UC Berkeley
joint with Yuji Kodama

Plan of the talk

- Background on the real Grassmannian and the KP equation
- Soliton solutions: a tropical approximation
- Asymptotics of solitons and the positroid stratification of the Grassmannian
- Asymptotics of solitons and the Deodhar decomposition of the Grassmannian
- Total positivity, cluster algebras, and soliton solutions
- Connection with triangulations

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References

- KP solitons, total positivity, and cluster algebras (Kodama + Williams), PNAS, May 2011.
- KP solitons and total positivity on the Grassmannian (K. + W.), `http://front.math.ucdavis.edu/1106.0023`.
- The Deodhar decomposition of the Grassmannian and the regularity of KP solitons
(K. + W.), `http://front.math.ucdavis.edu/1204.6446`.
- Network parameterizations of the Grassmannian (Talaska + W.), `http://front.math.ucdavis.edu/1210.5433`.

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The real Grassmannian

The Grassmannian $Gr_{kn} = Gr_{kn}(\mathbb{R}) = \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$.

Represent an element of $Gr_{kn}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

Given $I \in \binom{[n]}{k}$, the Plücker coordinate $\Delta_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The *Plücker embedding* of $Gr_{kn}(\mathbb{R})$ is the map $Gr_{kn}(\mathbb{R}) \to \mathbb{P}^{\binom{n}{k}-1}$ which sends

$$A \mapsto (\Delta_I(A))_{I \in \binom{n}{k}}.$$

Example

$$\text{Let } A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

$$\text{Then } A \mapsto (1 : c : d : -a : -b : ad - bc) \in \mathbb{P}^5.$$

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The Kadomtsev-Petviashvili equation

The KP equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0$$

- Proposed by Kadomtsev and Petviashvili in 1970, in order to study the stability of the one-soliton solution of the Korteweg-de Vries (KdV) equation under the influence of weak transverse perturbations.
- Studied by Sato, Hirota, Freeman-Nimmo, many many others ...

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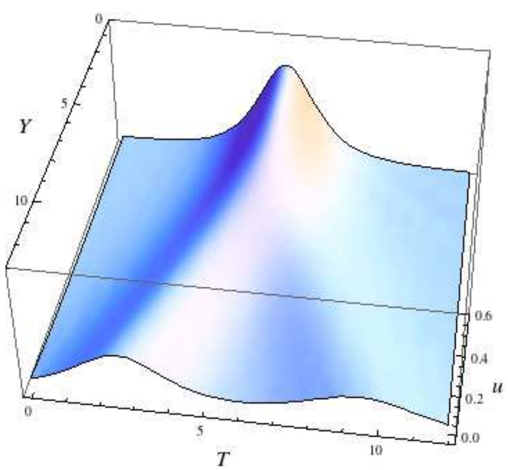

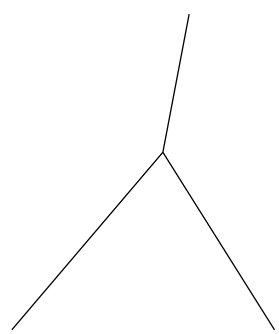

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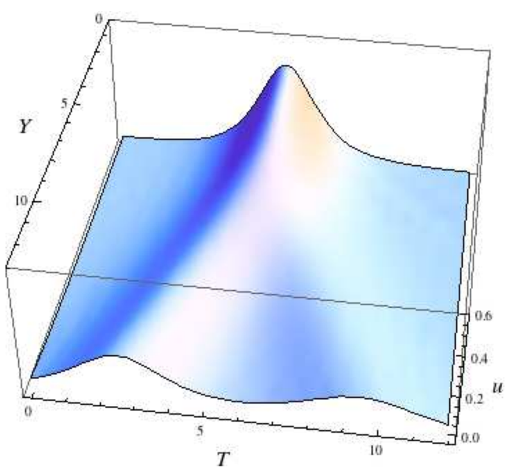

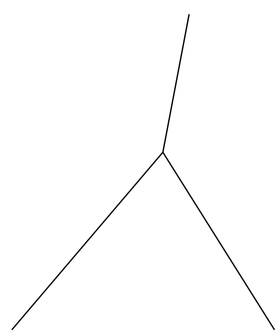

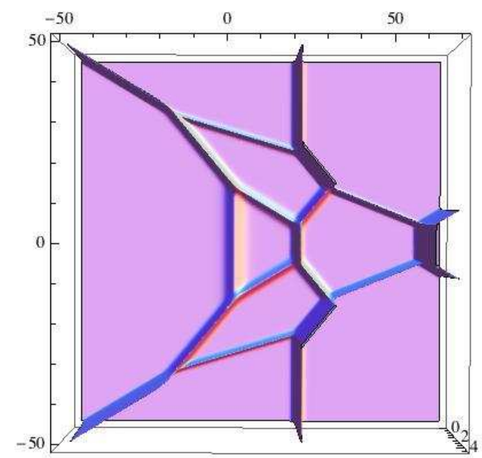

Soliton solutions to the KP eqn and shallow water waves

In 1834, John Scott Russell (a Scottish naval engineer) described:

"I was observing the motion of a boat . . . when the boat suddenly stopped – but not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the *Wave of Translation*."

Soliton solutions to the KP equation

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From $A \in Gr_{kn}(\mathbb{R})$, can construct τ_A, and then a solution u_A of the KP equation.
(cf Sato, Hirota, Satsuma, Freeman-Nimmo, Kodama, Chakravarty ...)

The τ-function τ_A

Fix real parameters κ_j such that $\kappa_1 < \kappa_2 < \cdots < \kappa_n$.
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A solution $u_A(x,y,t)$ of the KP equation (Freeman-Nimmo)

Choose $A \in Gr_{kn}(\mathbb{R})$, choose $\kappa_1 < \cdots < \kappa_n$, define $\tau_A(x,y,t)$ as above.

$$\text{Then } u_A(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \tau_A(x,y,t) \quad \text{is a solution to KP.}$$

Note: Whenever $\tau_A(x,y,t) = 0$, $u_A(x,y,t)$ will have a singularity.

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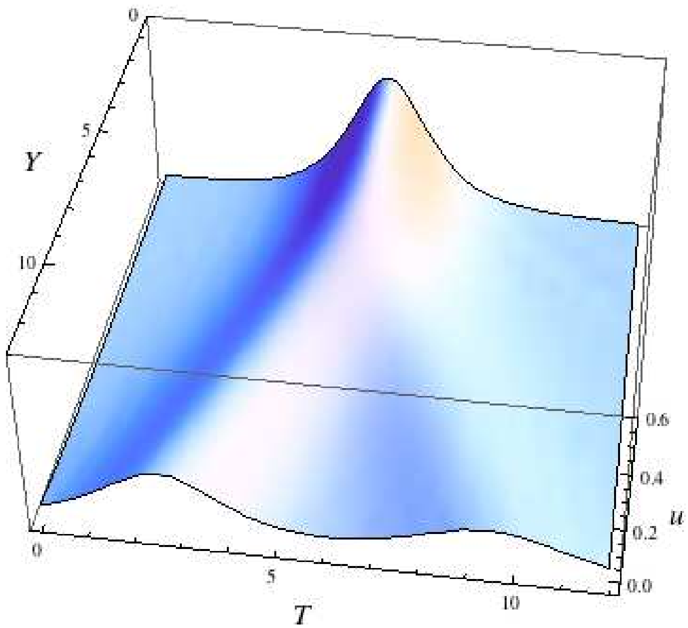
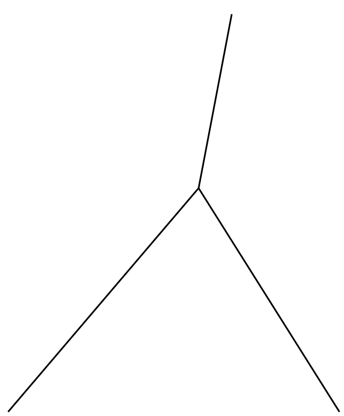
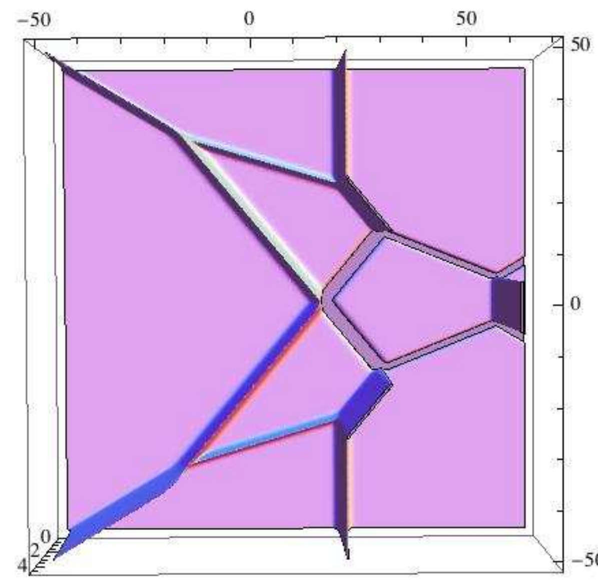
The contour plot of $u_A(x, y, t)$

We analyze $u_A(x, y, t)$ by fixing t, and drawing its *contour plot* $\mathcal{C}_t(u_A)$ for fixed times t – this will approximate the subset of the xy plane where $|u_A(x, y, t)|$ takes on its maximum values or is singular.







Definition of the contour plot at fixed time t

$u_A(x,y,t)$ is defined in terms of $\tau_A(x,y,t) := \sum_{I \in \binom{[n]}{k}} \Delta_I(A) E_I(x,y,t)$.

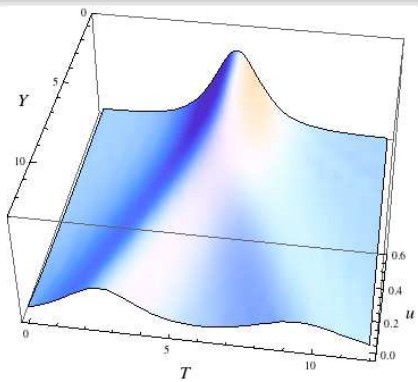
At most points (x,y,t), $\tau_A(x,y,t)$ will be dominated by one term –
– at such points, $u_A(x,y,t) \sim 0$.

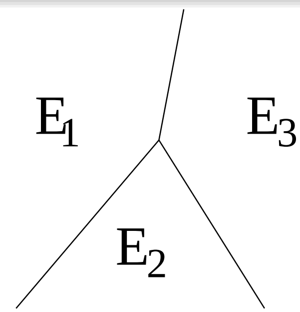
The *contour plot* $\mathcal{C}_t(u_A)$ is the subset of the xy plane where two or more terms dominate $\tau_A(x,y,t)$.

Approximates locus where $|u_A(x,y,t)|$ takes on max values or is singular.

We assume that x,y,t are on a large scale; then approximation is good.

When the κ_i's are integers, $\mathcal{C}_t(u_A)$ is a *tropical curve.*





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Label each region by the dominant exponential.

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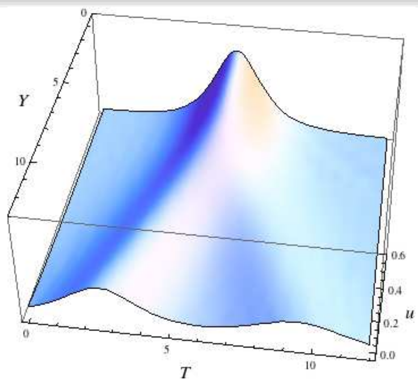

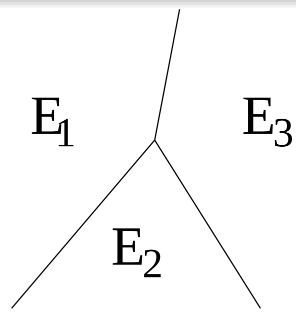

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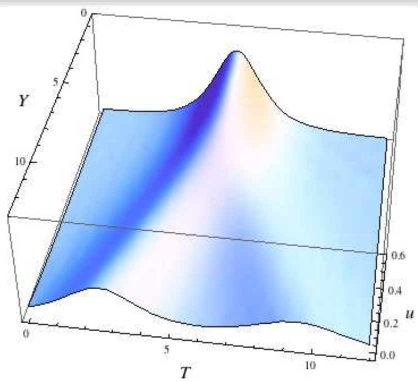

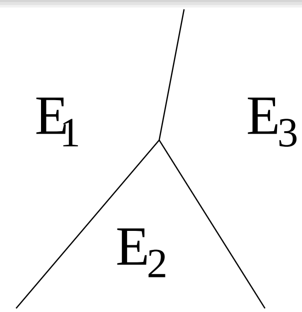

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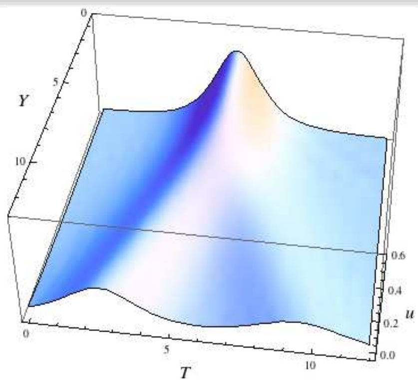

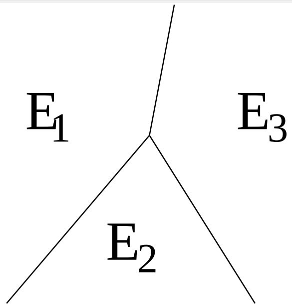

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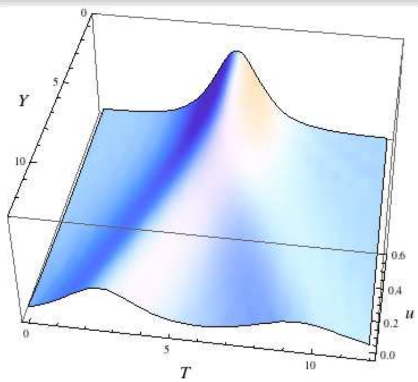

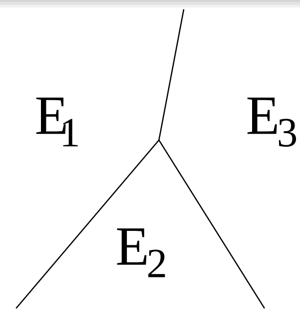

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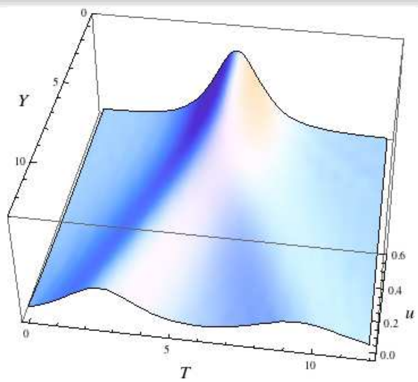

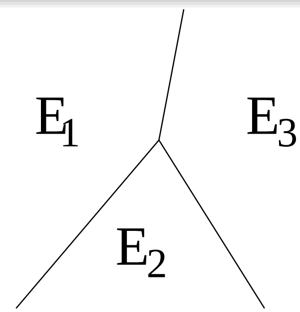

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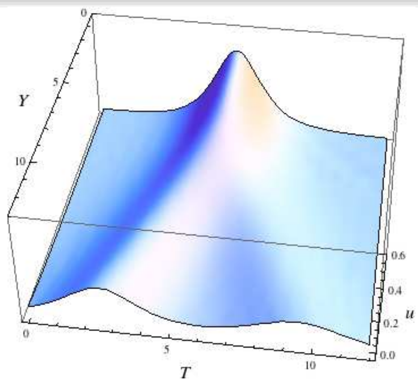

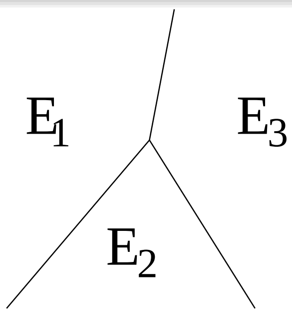

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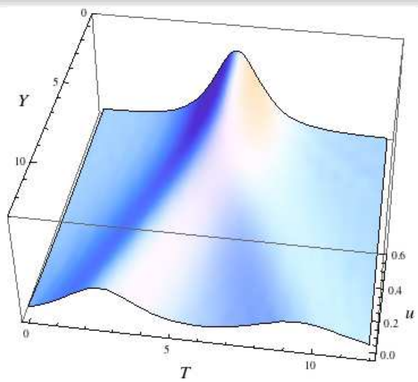

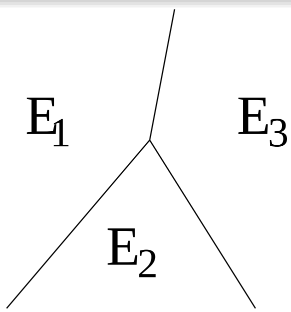

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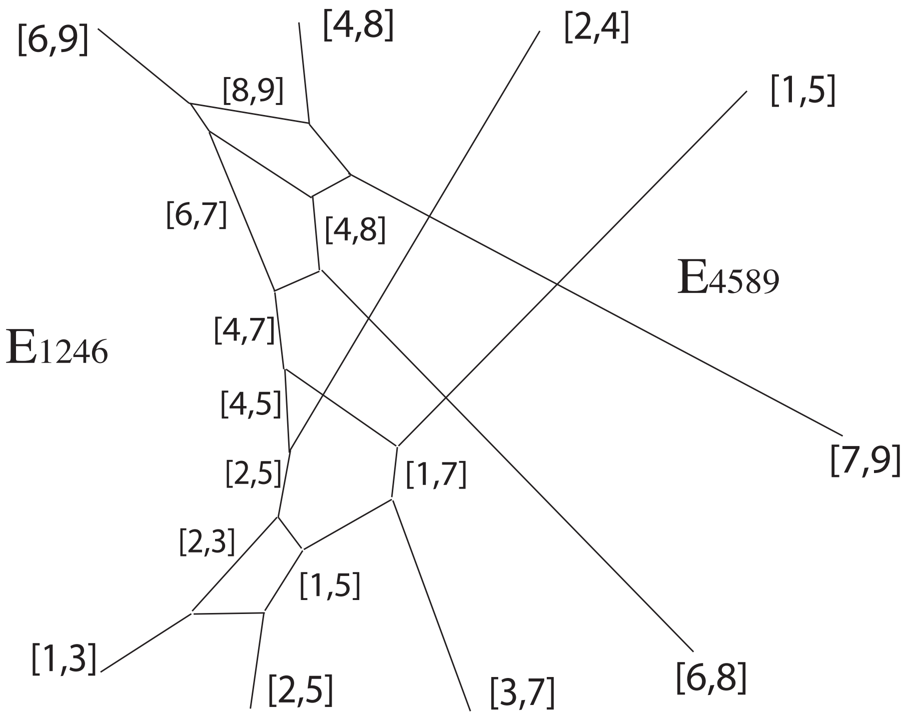
Visualizing soliton solutions to the KP equation

Generically, interactions of *line-solitons* are trivalent or are X-*crossings*.



If two adjacent regions are labeled E_I and E_J, then $J = (I \setminus \{i\}) \cup \{j\}$.
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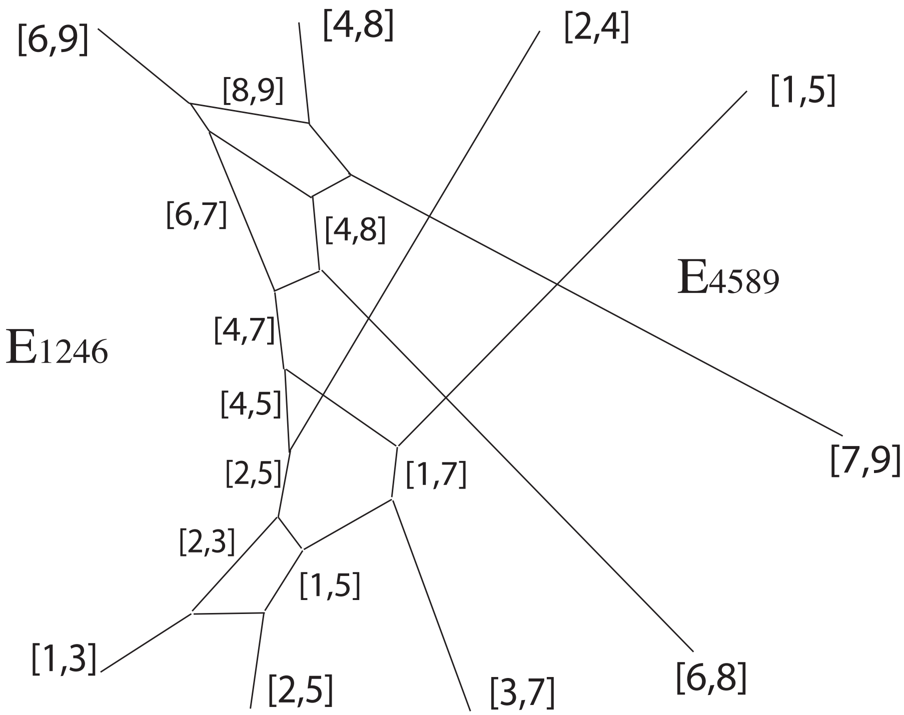

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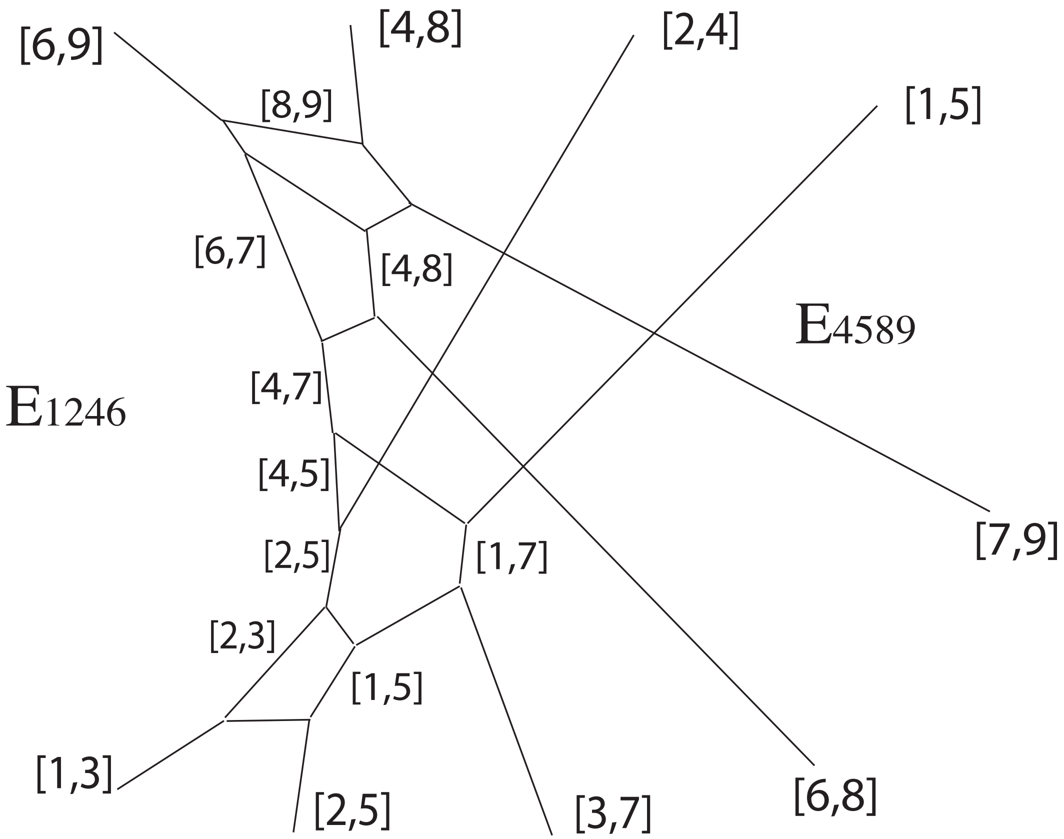
Extract combinatorial structure: Soliton graphs

We associate a *soliton graph* $G_t(u_A)$ to a contour plot $\mathcal{C}_t(u_A)$ by: forgetting lengths and slopes of edges, and marking a trivalent vertex black or white based on whether it has a unique edge down or up. Embed graph in disk.



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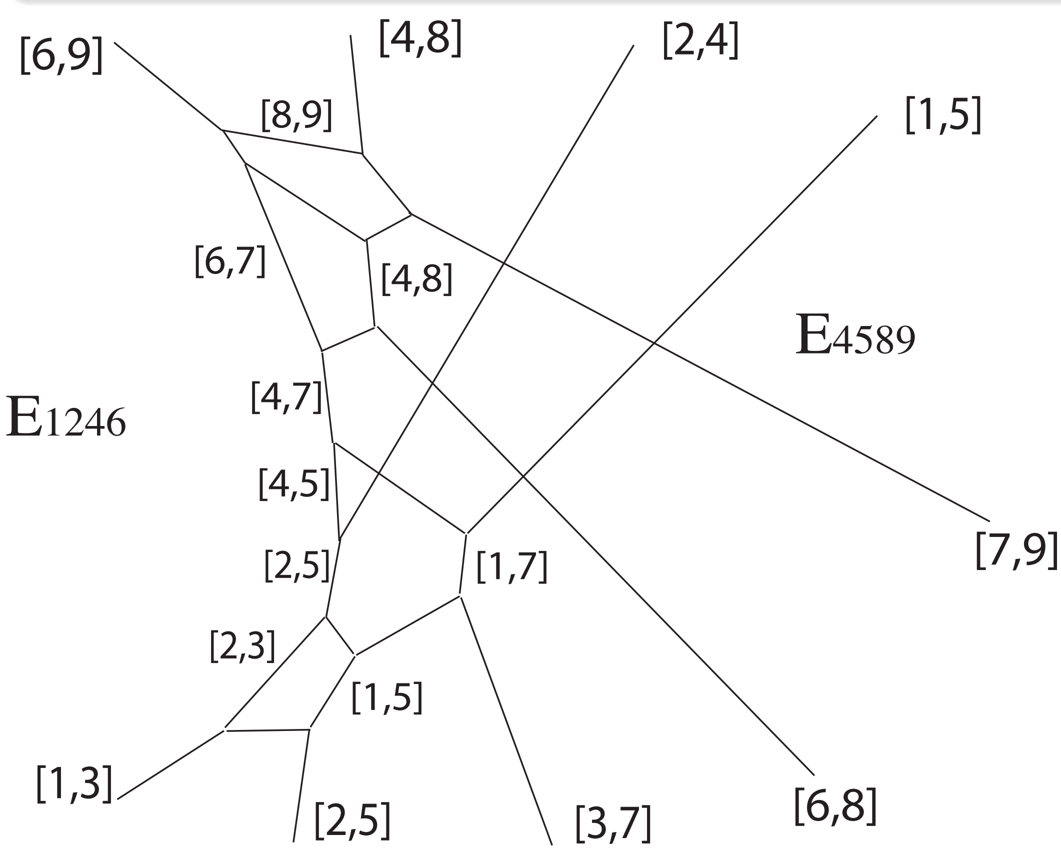
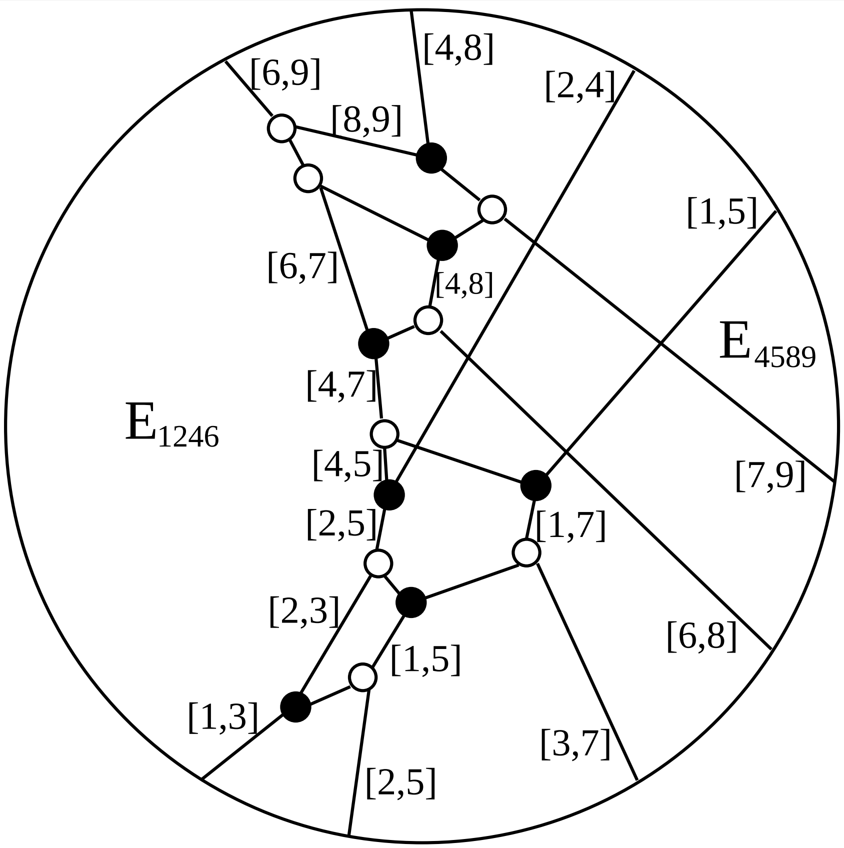
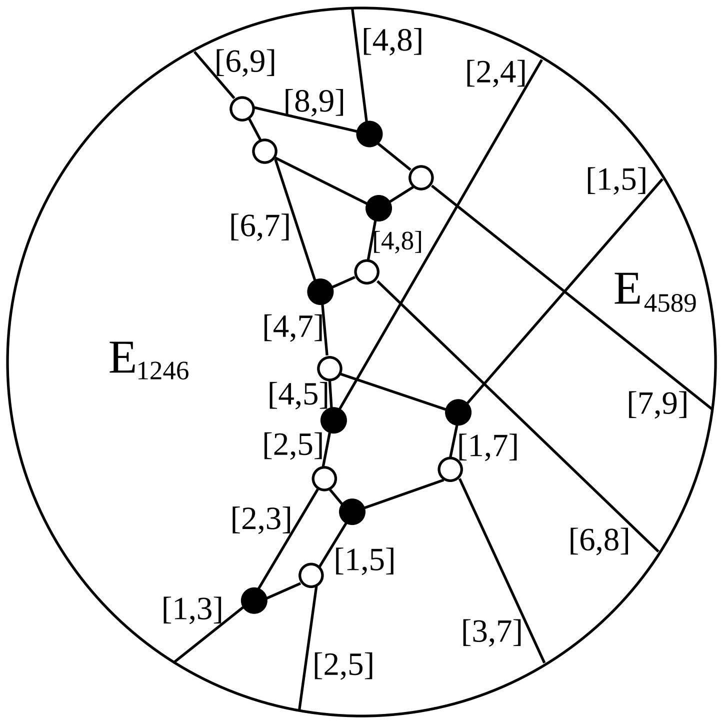




Soliton graph → bicolored graph



Associate a *bicolored graph* to each soliton graph by:

- For each unbounded line-soliton $[i, j]$ (with $i < j$) heading to $y >> 0$, label the incident boundary vertex by j.
- For each unbounded line-soliton $[i, j]$ (with $i < j$) heading to $y << 0$, label the incident boundary vertex by i.
- Forget the labels of line-solitons and regions.

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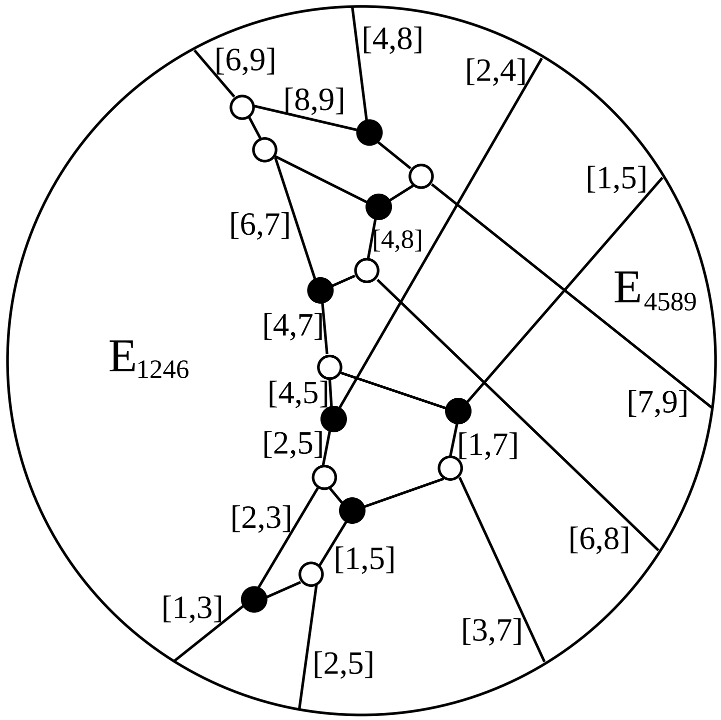

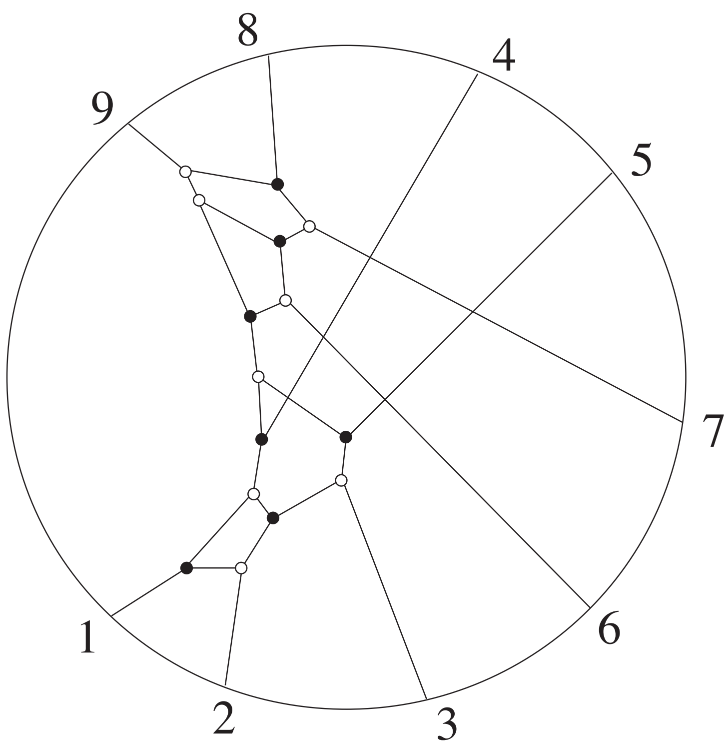

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Theorem. Passing from the soliton graph to the bicolored graph does not lose any information!

We can reconstruct the labels by following the "rules of the road" (zig-zag paths). From the bdry vertex i, turn right at black and left at white.

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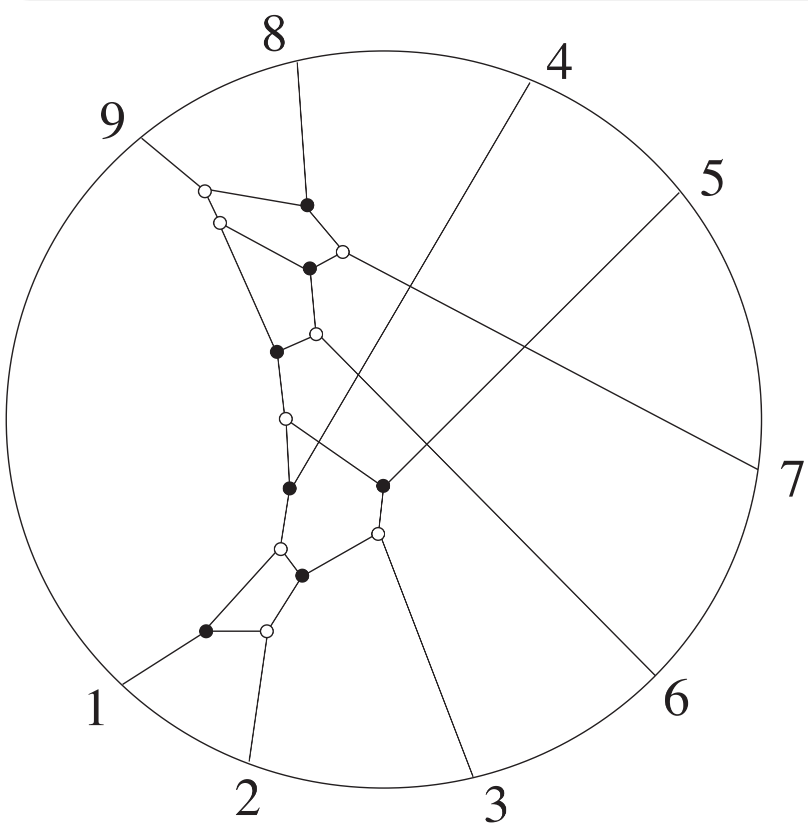

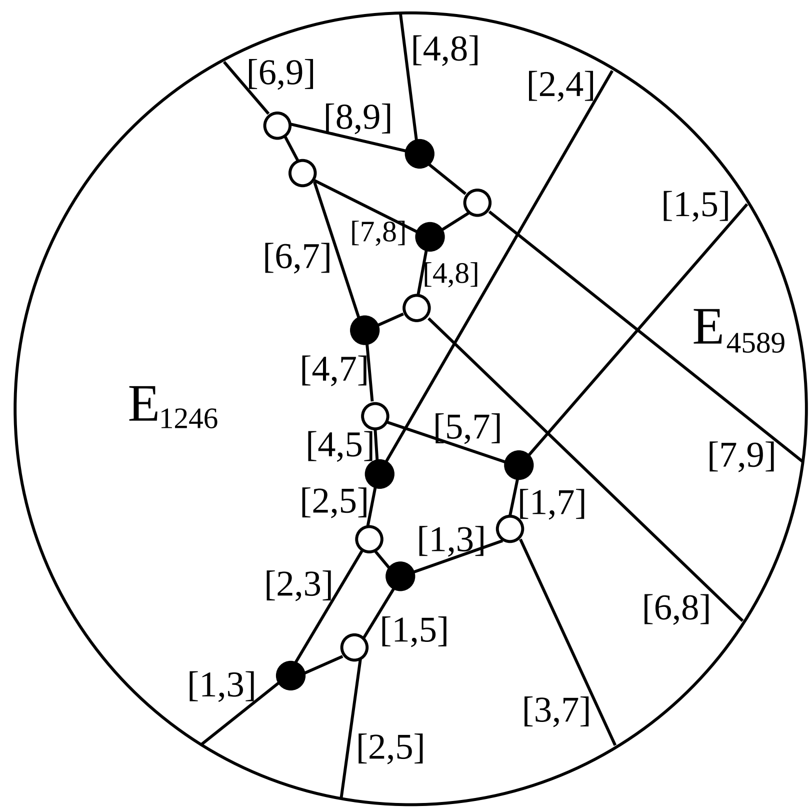

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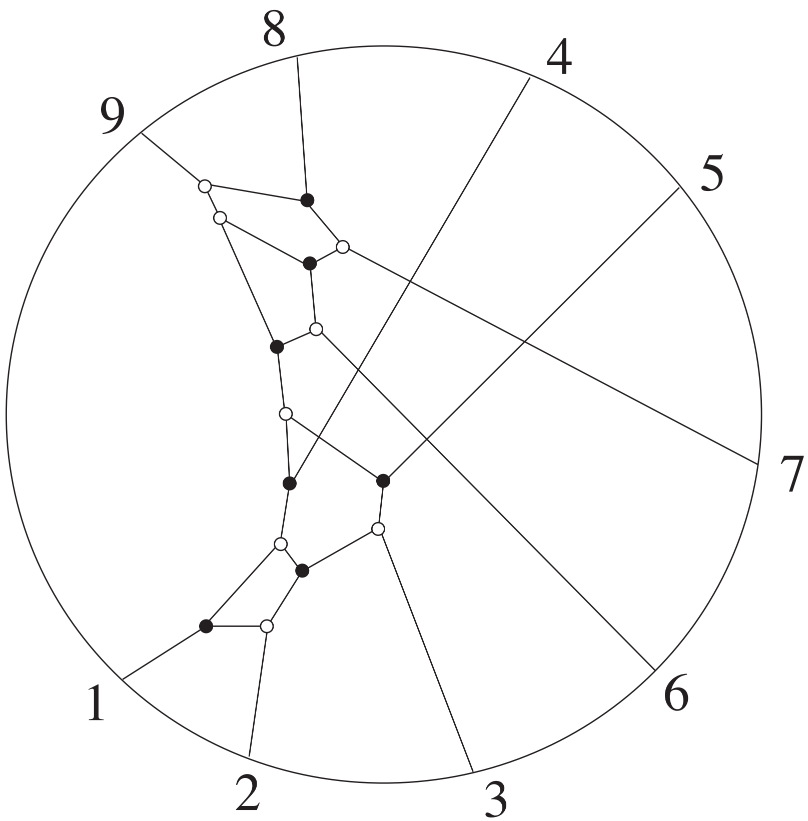

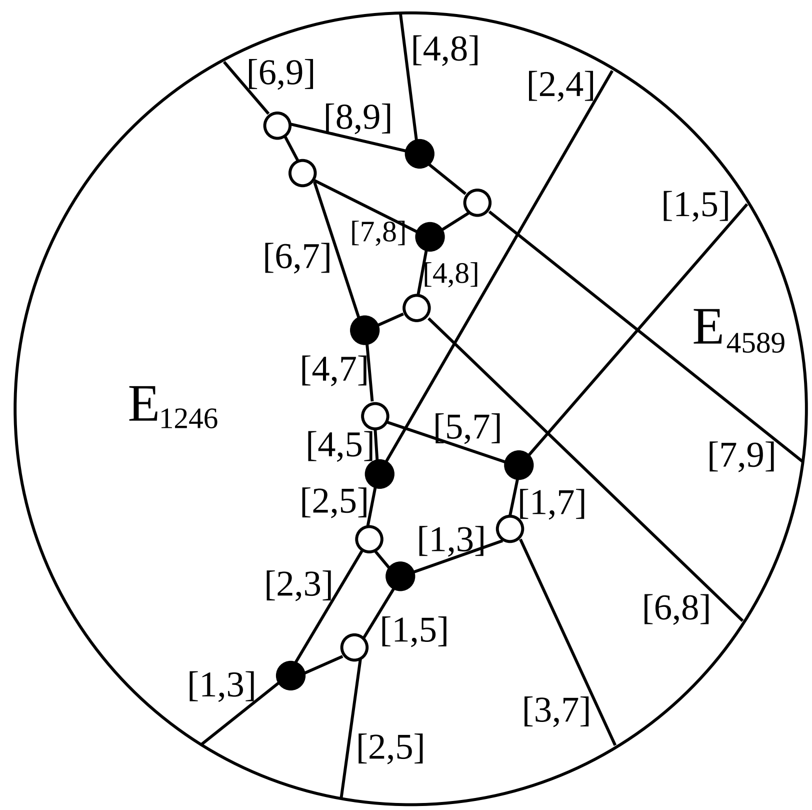

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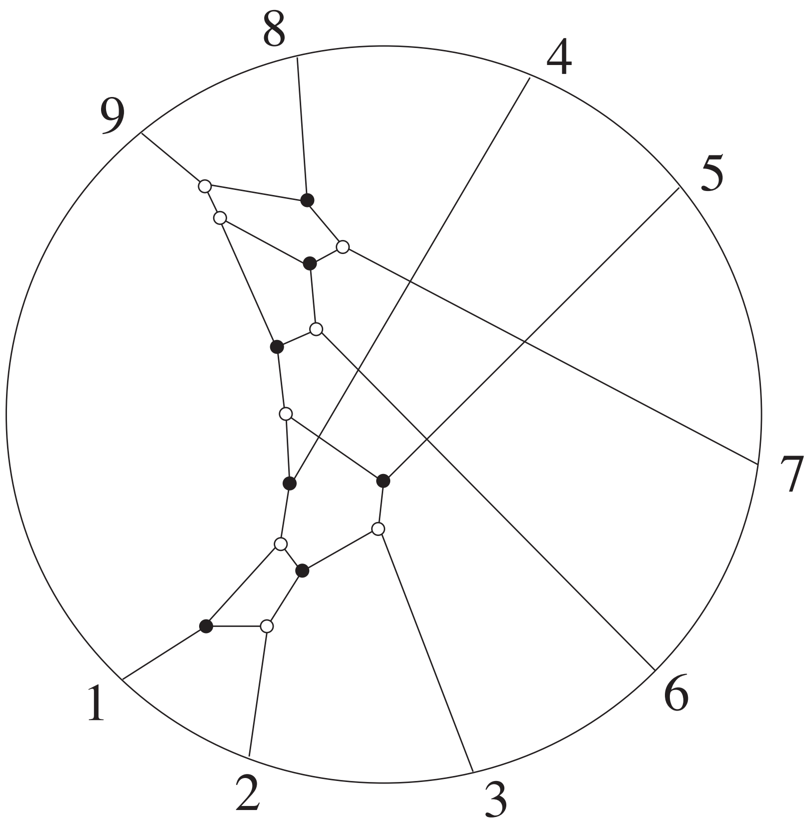

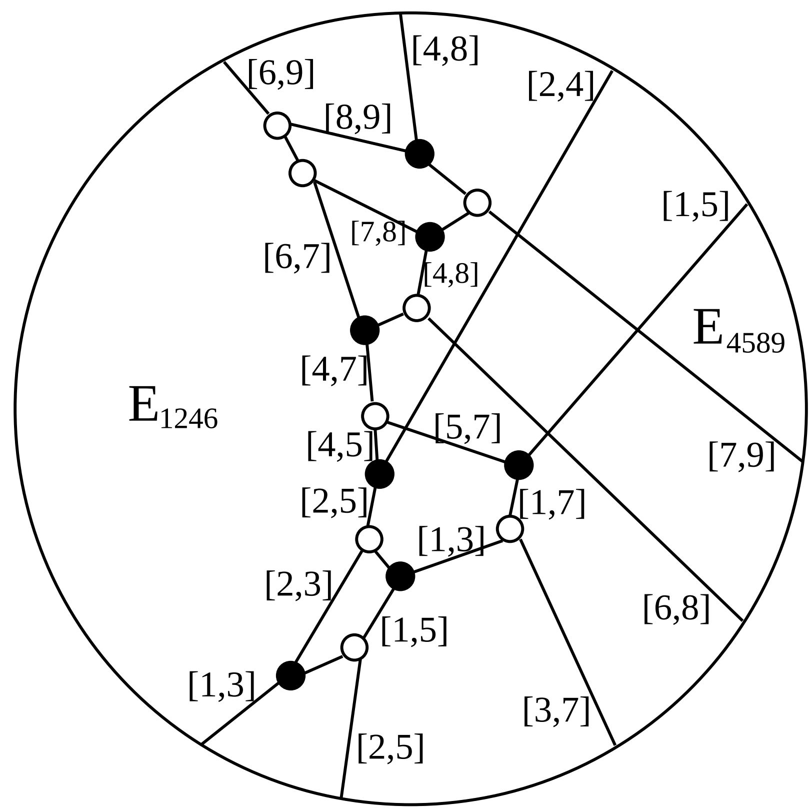

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Asymptotics of soliton solutions

Recall

Fix $A \in Gr_{kn}(\mathbb{R})$ and $\kappa_1 < \cdots < \kappa_n$. This gives rise to a soliton solution $u_A(x, y, t)$ of the KP equation.

How does $u_A(x, y, t)$ behave as y tends to $\pm\infty$?

This depends precisely on which *positroid stratum* A lies in.

How does $u_A(x, y, t)$ behave as t tends to $\pm\infty$?

This depends precisely on which *Deodhar component* A lies in.

Some decompositions of the Grassmannian

Matroid strat. $\prec$ Deodhar dec. $\prec$ Positroid strat. $\prec$ Schubert dec.

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Asymptotics of soliton solutions when $y \to \pm\infty$

Given any $A \in Gr_{kn}(\mathbb{R})$, we can completely determine the *asymptotics* (as $y \to \pm\infty$) of the contour plot $\mathcal{C}_t(u_A)$, using Postnikov's *positroid stratification* of the real Grassmannian.

A *Grassmann necklace* of type (k, n) is a sequence $\mathcal{I} = (I_1, \dots, I_n)$ of k-element subsets of $[n] = \{1, 2, \dots, n\}$ such that:

- For $i \in [n]$, if $i \in I_i$, then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$.
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Example: (134, 345, 345, 456, 356, 346).

A *decorated permutation* is a permutation of $\{1, 2, \dots, n\}$ such that fixed points are colored in one of two colors. In bijection with Grass. necklaces!

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- For $i \in [n]$, if $i \in I_i$, then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$.
- If $i \notin I_i$, then $I_{i+1} = I_i$.

Example: (134, 345, 345, 456, 356, 346).

A *decorated permutation* is a permutation of $\{1, 2, \dots, n\}$ such that fixed points are colored in one of two colors. In bijection with Grass. necklaces!

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$$\text{Example: } A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \end{pmatrix} \mapsto (12, 24, 34, 42).$$

Given a Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ of type (k, n), the *positroid stratum* $S_{\mathcal{I}}$ is the set of elements of Gr_{kn} whose Grassmann necklace is $\mathcal{I}$.

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Asymptotics of soliton solutions at $y \to \pm\infty$

Grassmann necklaces $\mathcal{I} \leftrightarrow$ dec. perms $\pi \leftrightarrow$ Positroid strata $S_{\mathcal{I}} = S_\pi$.

An *excedance* of a permutation π is a position i such that $\pi(i) > i$.
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Theorem (Kodama-W., generalizes Chakravarty-Kodama)

Let $A \in S_\pi$. Consider the contour plot $\mathcal{C}_t(u_A)$ at any time t:

- *its line-solitons at $y >> 0 \leftrightarrow$ the excedances $[i, \pi(i)]$ of π*
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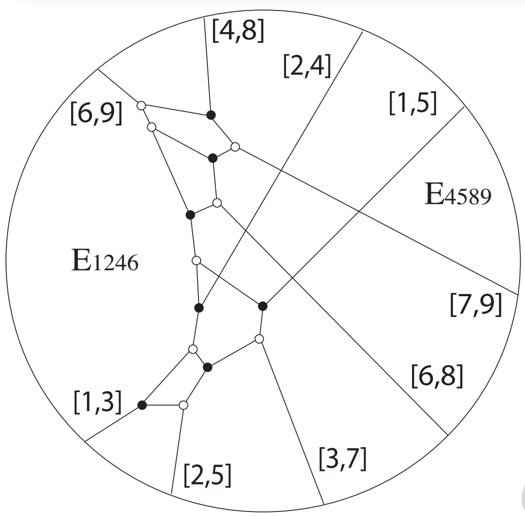

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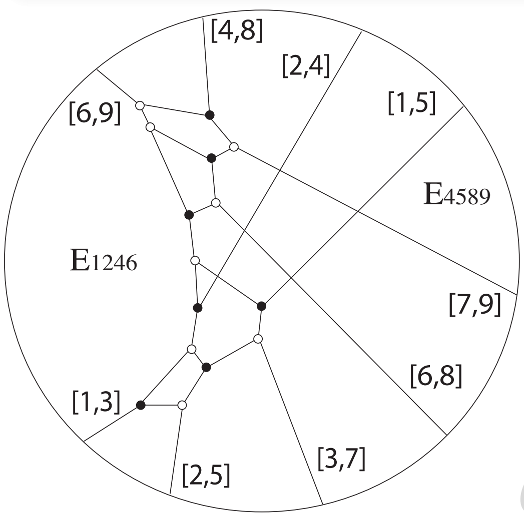

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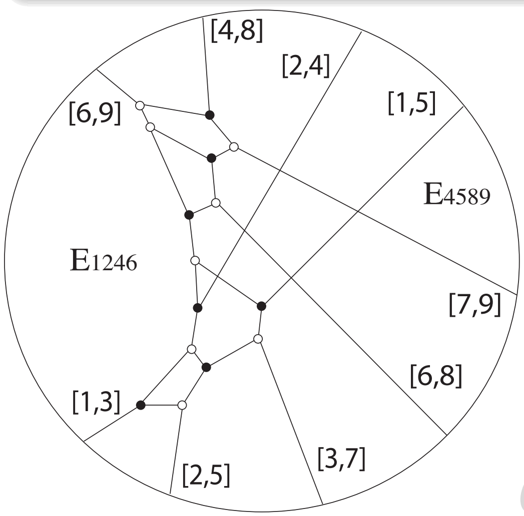

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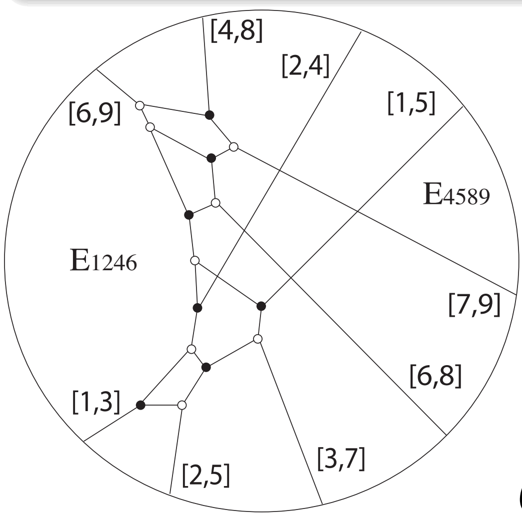

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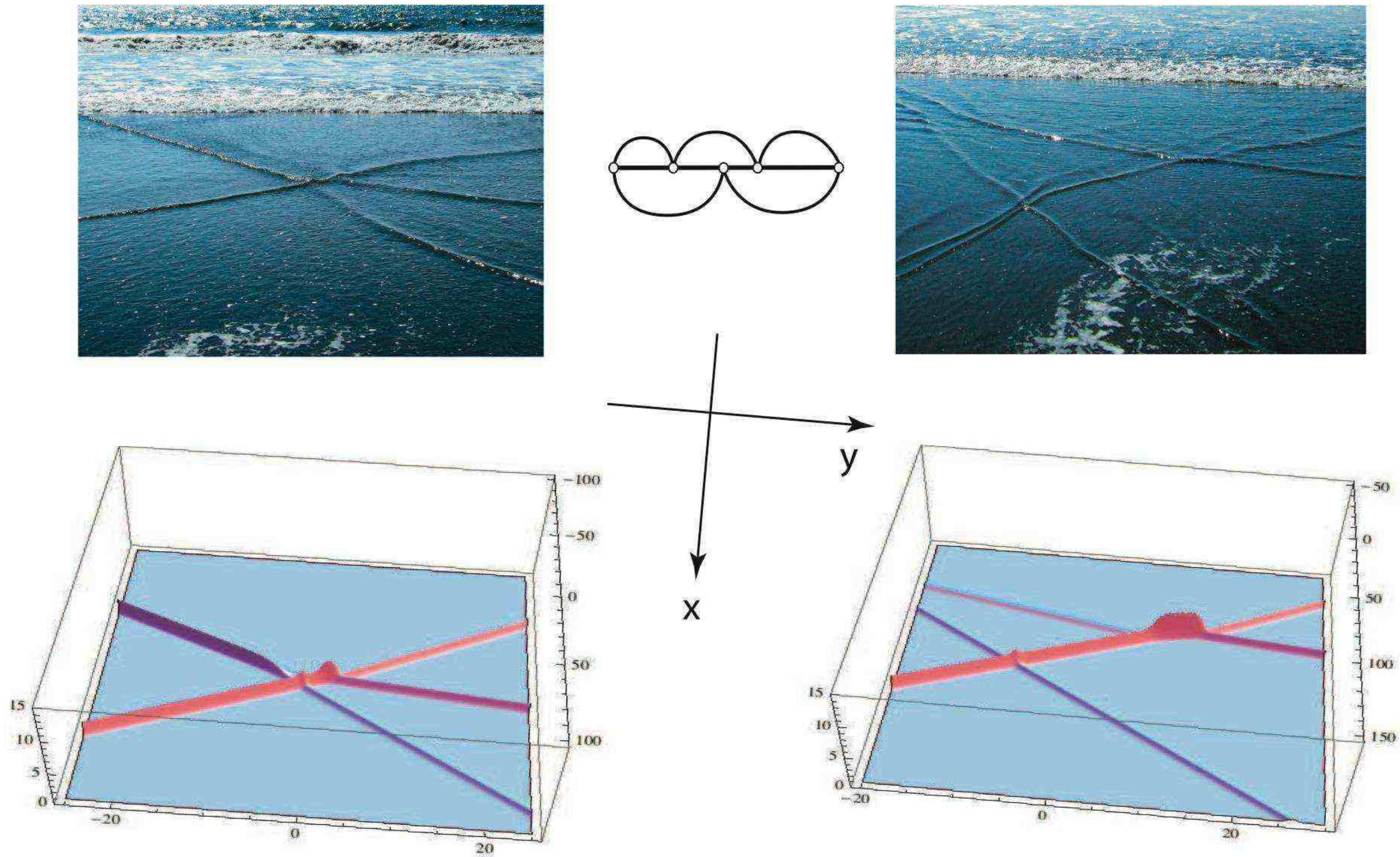

(Photos due to Mark Ablowitz.)

Asymptotics of soliton solutions when $t \to \pm\infty$

Put each $A \in Gr_{kn}$ in *row-echelon form*, i.e. $A = \begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix}$.

Let $\pi_i(A) \in Gr_{i,n}$ be the projection of A to the bottom i rows, $1 \leq i \leq k$. Each $\pi_i(A)$ lies in some positroid stratum $S_{\mathcal{I}^i} \subset Gr_{i,n}$.

By considering the collection $S_{\mathcal{I}^1}, S_{\mathcal{I}^2}, \dots, S_{\mathcal{I}^k}$, and using our results on the soliton asymptotics at $y \to \pm\infty$, we can inductively determine the contour plot $C_t(u_A)$ for $t << 0$.

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Combinatorics of the Deodhar decomposition of Gr_{kn}

Deodhar components are in bijection with tableaux we call *Go-diagrams*. They represent factorizations (not necessarily reduced) of permutations.

A filling of a Young diagram with empty boxes, and black and white stones, is a *Go-diagram* if the corresponding *wiring diagram* satisfies:

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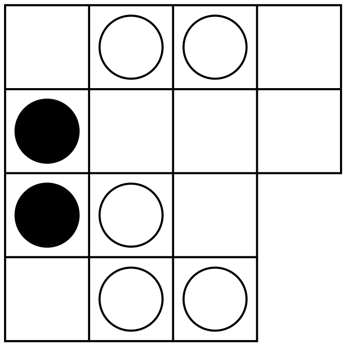

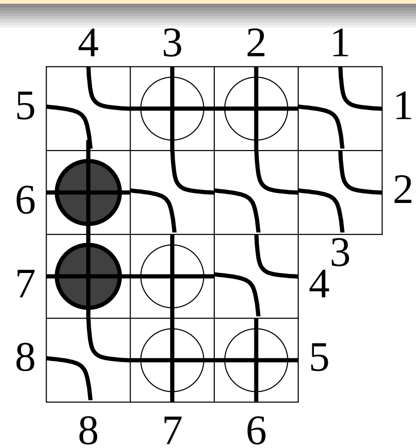

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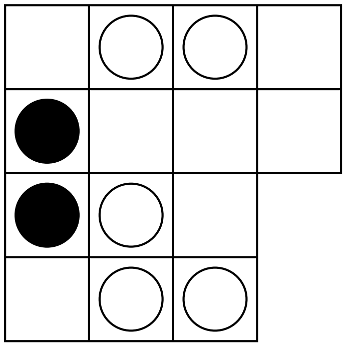

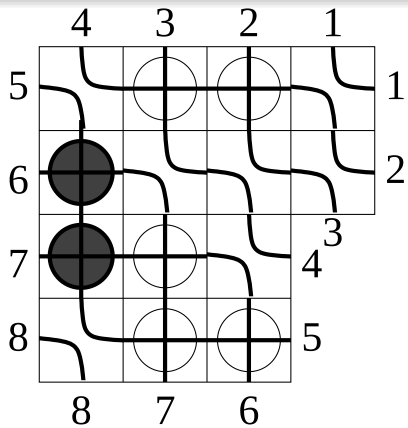

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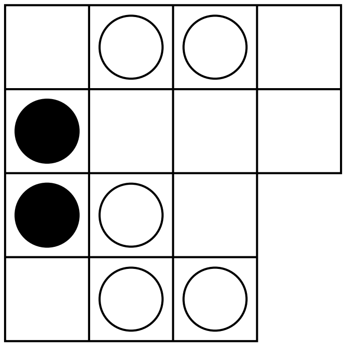

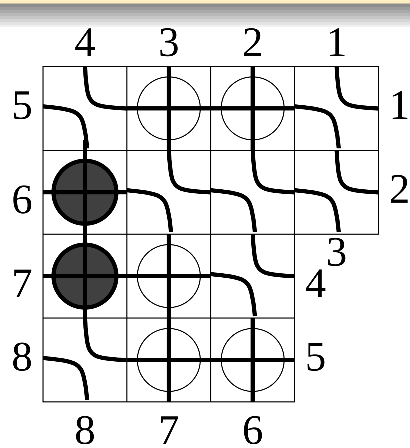

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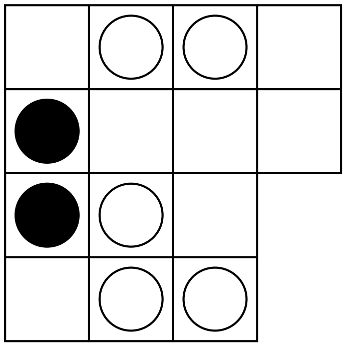

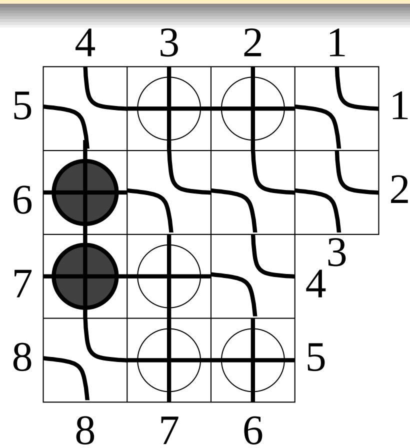

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- When one follows the wires from southeast to northwest, each time two wires have the opportunity to cross so as to decrease the length of the permutation, they must cross.
- White (resp. black) stones represent crossings which increase (resp. decrease) the length of the permutation.

The Deodhar component determines $t << 0$ asymptotics

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Let D be a Go-diagram. The following procedure realizes the soliton graph $G_t(u_A)$ for any A in the Deodhar component S_D when $t << 0$.

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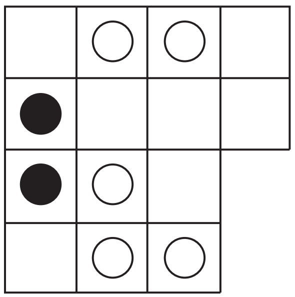

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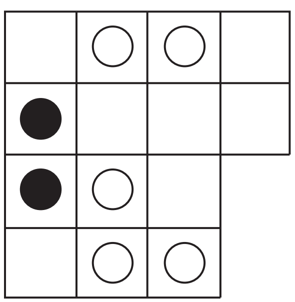

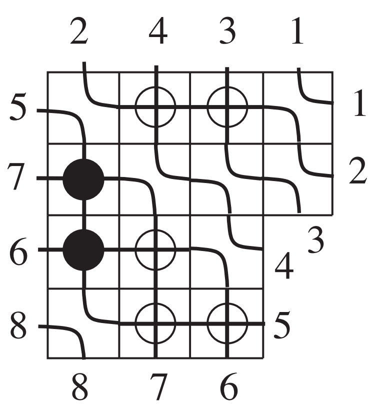

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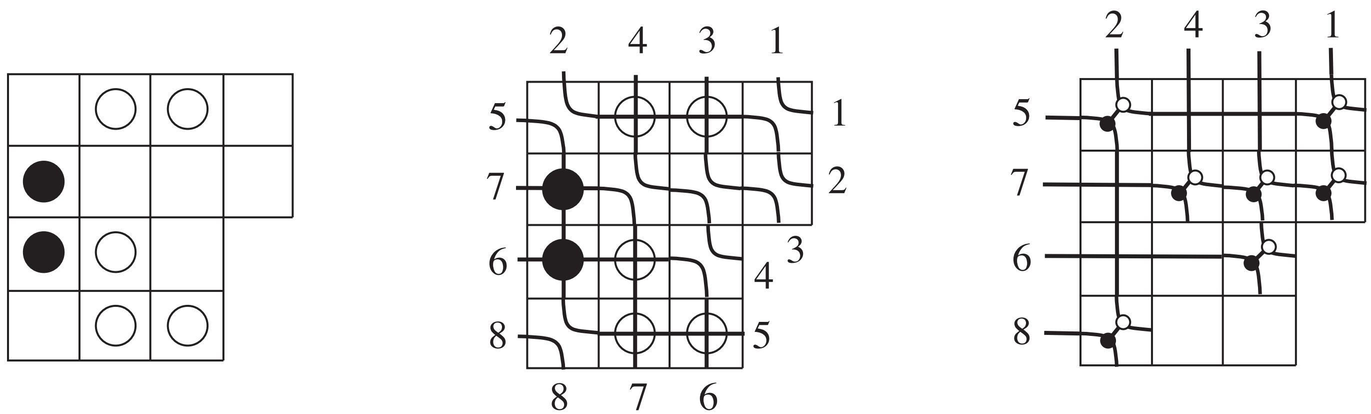

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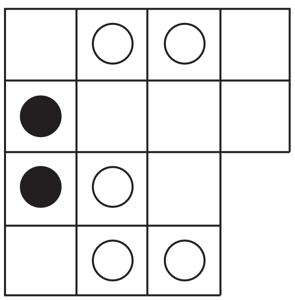

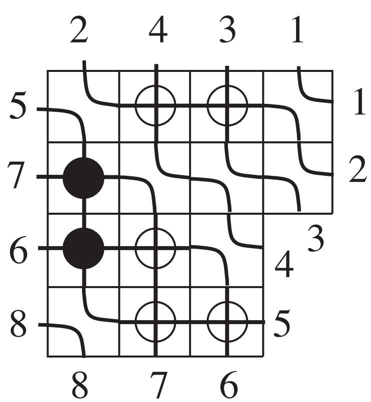

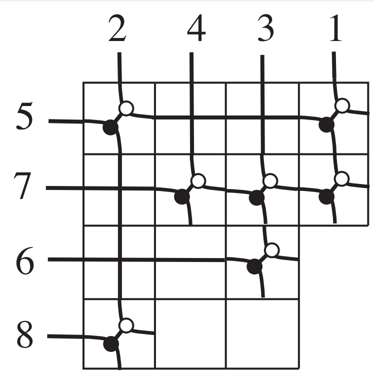

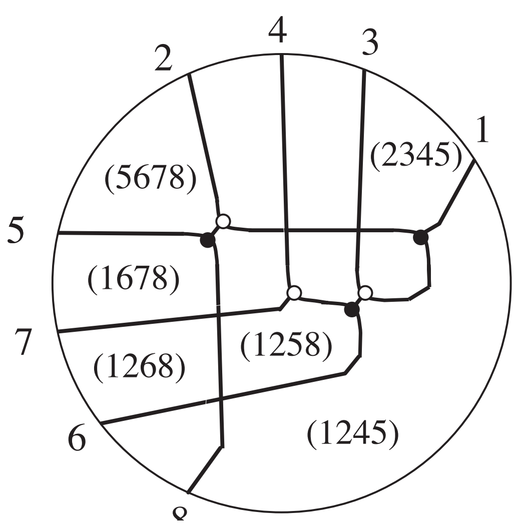

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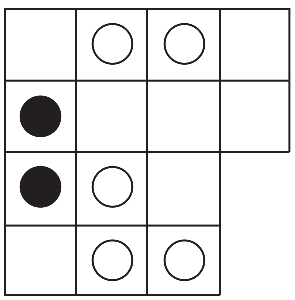
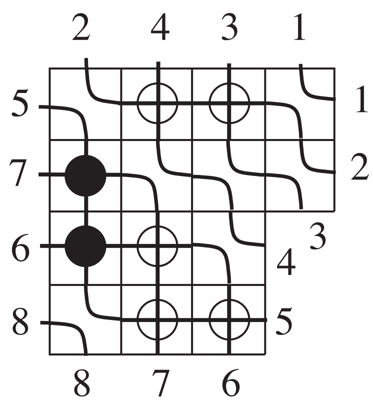
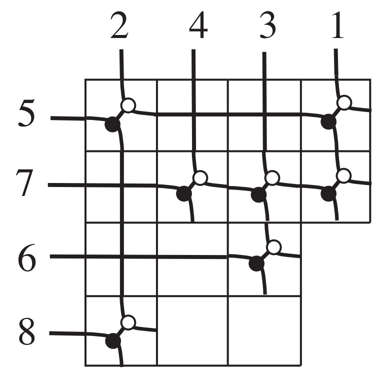
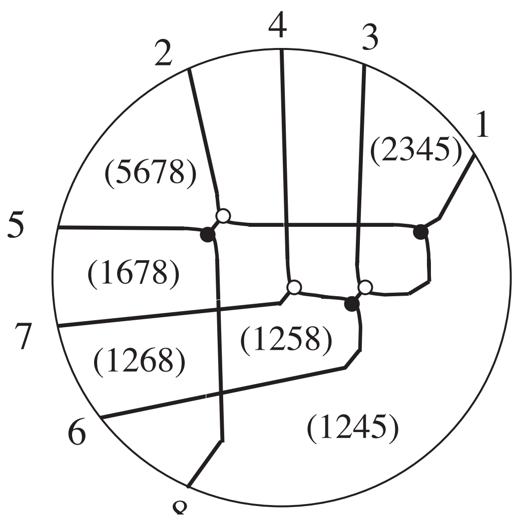
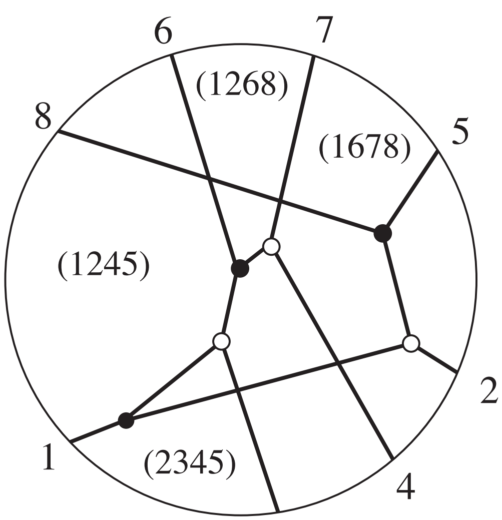




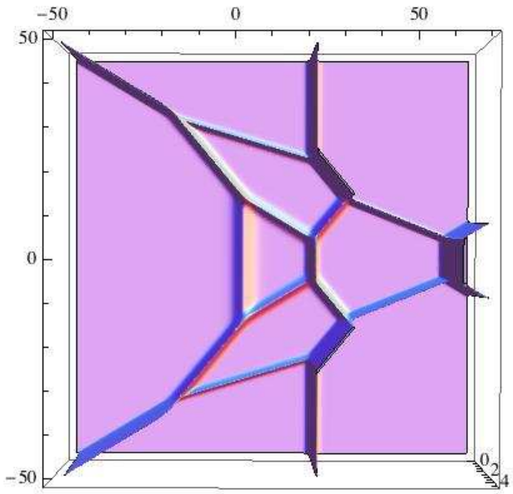


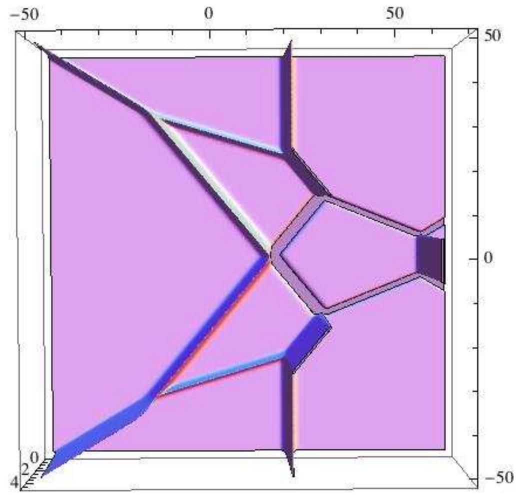


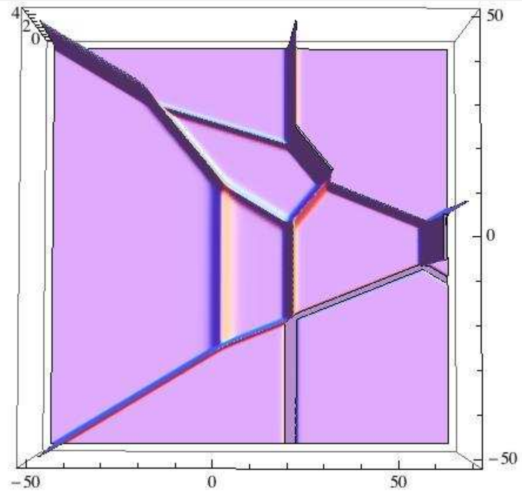


The regularity problem for KP solitons

Given $A \in Gr_{kn}$, when will $u_A(x, y, t)$ be *regular* for all x, y, t?







Recall: $\tau_A(x, y, t) := \sum_{J \in \binom{[n]}{k}} \Delta_J(A) E_J(x, y, t)$.

Then $u_A(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \tau_A(x, y, t)$ is a solution to KP.

Theorem

Given $A \in Gr_{kn}$, $u_A(x, y, t)$ is regular $\forall x, y, t$ iff each $\Delta_J(A) \geq 0$.

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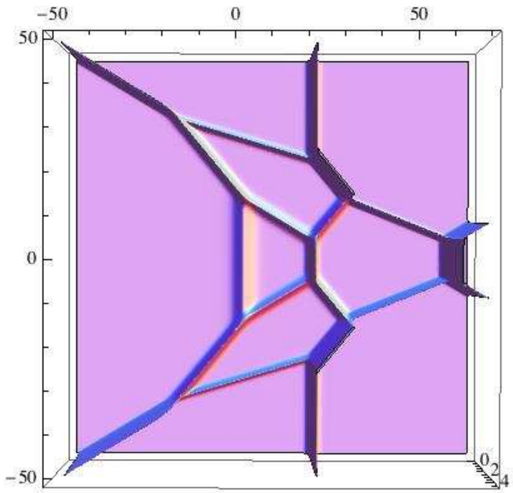
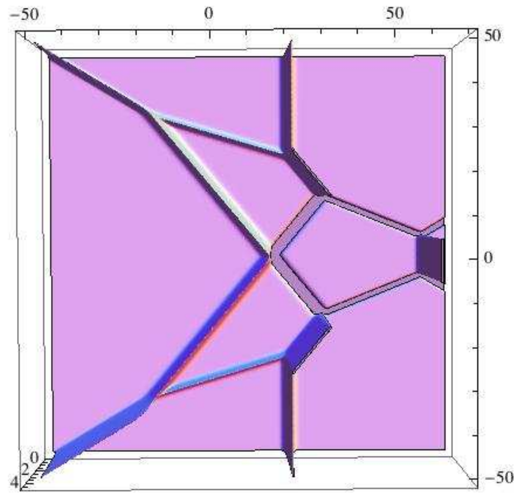
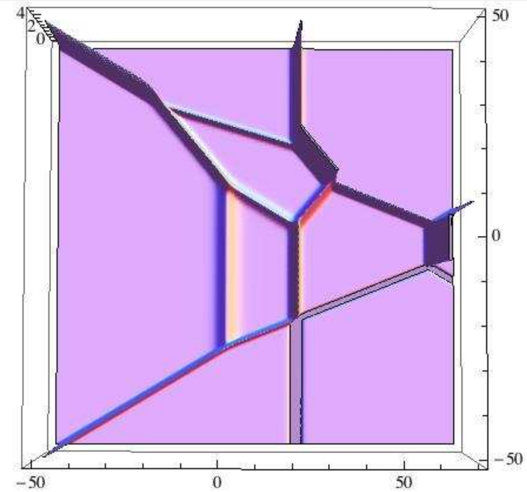

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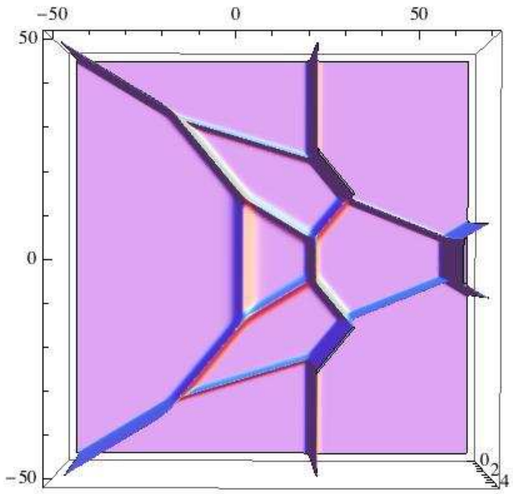

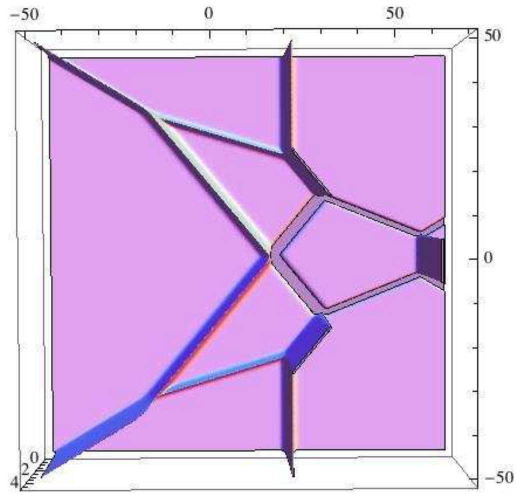

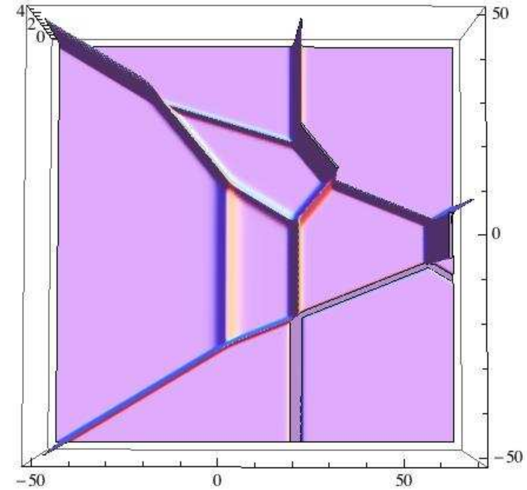

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Total positivity for the Grassmannian

The previous *regularity theorem* motivates the following definition.

The *totally non-negative part of the Grassmannian* $(Gr_{kn})_{\geq 0}$ is the subset of $Gr_{kn}(\mathbb{R})$ where all Plücker coordinates $\Delta_I(A) \geq 0$.

Similarly define $(Gr_{kn})_{>0}$ using $\Delta_I(A) > 0$.

Theorem (rephrased)

Given $A \in Gr_{kn}$, $u_A(x, y, t)$ is regular $\forall x, y, t$ if and only if $A \in (Gr_{kn})_{\geq 0}$.

Brief history of total positivity

1930's: Study of totally positive matrices, matrices with all minors > 0.

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- Connection with cluster algebras
- We can solve the inverse problem
- We can classify soliton graphs coming from $(Gr_{2n})_{>0}$.

Theorem

Let $A \in (Gr_{kn})_{>0}$. If $G_t(u_A)$ is generic (no vertices of degree > 3), then the set of dominant exponentials labeling $G_t(u_A)$ is a cluster for the cluster algebra associated to the Grassmannian.

To prove this, the main step is to show that $G_t(u_A)$ is a *reduced plabic graph.*[a] Then the result follows from Scott's work on the cluster algebra structure of $\mathbb{C}[Gr_{kn}]$.

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Let $A \in (Gr_{kn})_{\geq 0}$ and consider $u_A(x, y, t)$. Given t together with the contour plot of $u_A(x, y, t)$, can one reconstruct the point of $(Gr_{kn})_{\geq 0}$ which gave rise to the solution?

Theorem

1. For $t << 0$, we can always solve the inverse problem.
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Proof of 1: uses our description of soliton graphs at $t << 0$, and work of Kelli Talaska.

Proof of 2: uses our result that the set of dominant exponentials labeling such a contour plot forms a cluster for $\mathbb{C}[Gr_{kn}]$.

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Up to graph-isomorphism,[a] the generic soliton graphs for $(Gr_{2,n})_{>0}$ for all t are in bijection with triangulations of an n-gon. Therefore the number of different soliton graphs is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

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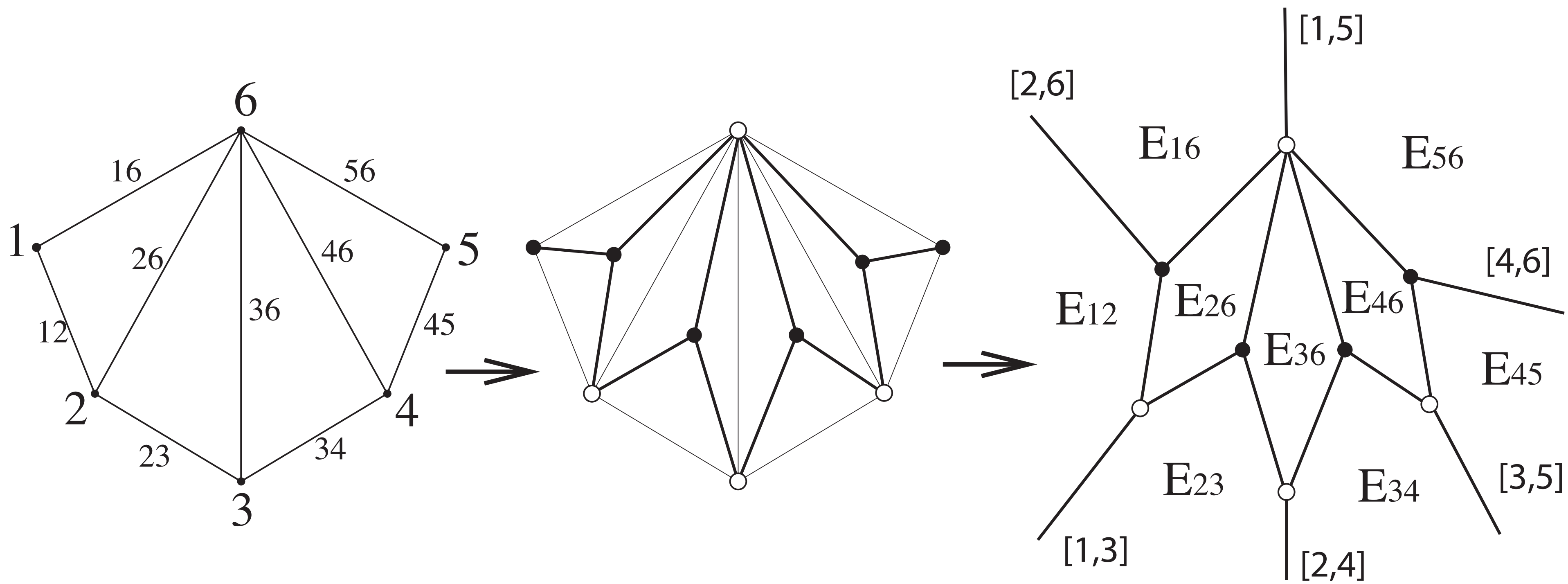

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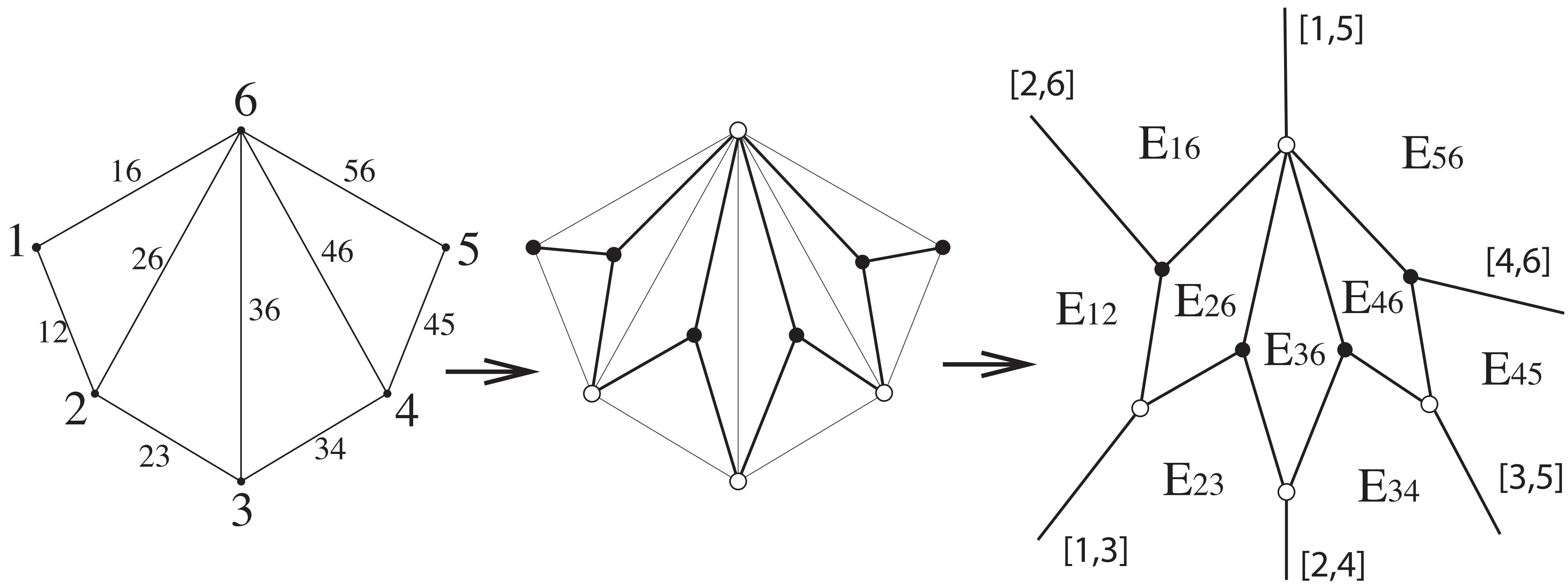
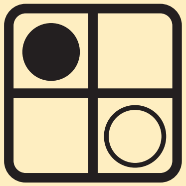


Conclusion

Haiku summary of my talk

Arrangements of stones
reveal patterns in the waves
as space-time expands

K.W.



とき超えて
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