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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SRHERMES@BRANDEIS.EDU

Speaker's Name: C. Geiss

Talk Title: On Generic Bases for Cluster Algebras

Date: 11 / 1 / 12 **Time:** 9 : 30 (am) / pm (circle one)

List 6-12 key words for the talk: Cluster Algebras, Semicanonical Bases, Quiver Varieties, Kac-Moody Groups, CC-functions, Potentials

Please summarize the lecture in 5 or fewer sentances: The speaker introduced the notion of generic CC-functions for the Jacobian algebra of a quiver w/ potential. He showed that for certain unipotent cells the coordinate ring is a cluster algebra and the generic CC-functions provide the dual canonical basis. Extended this technique to surface cluster algebras.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

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On Generic Bases for Cluster Algebras

C. Geiss

November 1, 2012

Plan:

1. Generic CC-functions (Caldero-Chapoton) and dual semicanonical basis (w/ Leclerc, Schröer [GLS])
2. Generic bases and bangle for surface algebras (w/ Lambardini, Schröer [GLaS])

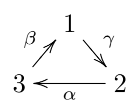
Notation: Let $\widetilde{Q}$ be an ice quiver and Q the full mutable subquiver. $Q_0\{1,\ldots,r\}$ and $\widetilde{Q}_0 = Q_0 \cup \{r+1,\ldots,s\}$. The cluster algebra $A(\widetilde{Q},\mathbf{x}) \subset \mathbb{Z}[x_1^{\pm},\ldots,x_s^{\pm}]$ (Laurent polynomials).

Aim: Geometric construction of a basis for $A(\widetilde{Q},\mathbf{x})$ which includes the cluster monomial. Geometric here means in the sense of [DWZ].

Find a non-degenerate (polynomial) potential W for Q. Assume $\Lambda = \mathbb{C}Q/(\partial W) = \widehat{\mathbb{C}Q}/\overline{(\partial W)}$ (true in the case considered today but not true in general).

Example 0.1. For the quiver

$Q =$



the potential $W = \alpha\beta\gamma$ is such a non-degenerate potential.

1 Generic CC-Functions

For $\mathbf{d} \in \mathbb{N}_0^r$ a dimension vector let $\mathsf{mod}_{\mathbf{d}}(\Lambda)$ be the variety of $\mathbf{d}$-dimensional representations of Λ. The group $\mathrm{GL}_{\mathbf{d}}$ acts on $\mathsf{mod}_{\mathbf{d}}(\Lambda)$ by conjugation so that the orbits are the isomorphism classes of representations. Let

$$\mathsf{Irr}(\Lambda) = \coprod_{\mathbf{d}} \mathsf{Irr}_{\mathbf{d}}(\mathsf{mod}_{\mathbf{d}}(\Lambda))$$

be the irreducible components.

Definition 1.1. A representation $Z \in \mathsf{Irr}(\Lambda)$ is *strongly reduced* iff $\operatorname{codim}_Z \mathrm{GL}_{\mathbf{d}} X = \dim \operatorname{Hom}_\Lambda(\tau^- X, X)$ (where x means X as a point in the variety).

Denote by $\mathsf{Irr}^{\mathrm{sr}}(\Lambda)$ the open subset of $\mathsf{Irr}(\Lambda)$ of strongly reduced representations.

Example 1.2. If $W = 0$ (e.g. Q acyclic) then every $\mathsf{mod}_{\mathbf{d}}(\Lambda)$ is irreducible. For the quiver in the first example there are no strongly irreducible modules with dimension vector 111.

Definition 1.3. The g-vector of X is defined to be $g_X = (\dim \operatorname{Ext}^1_\Lambda(S_i,X) - \dim \operatorname{Hom}_\Lambda(S_i,X))_{i=1}^r$.

Define

$$\hat{y}_k := \prod_{i=1}^{s} x_i^{\widetilde{Q}(i,k)-\widetilde{Q}(k,i)}$$

where $\widetilde{Q}(k,i)$ is the number or arrows k to i,

$$\phi_M := \mathbf{x}^{g_M} \sum_{\mathsf{k}} \chi(Gr^{\Lambda}_{\mathsf{k}}(M))\hat{y}^{\mathsf{k}},$$

and define ϕ_Z to be the generic value of $\phi_?$ of $Z \in \mathsf{Irr}(\Lambda)$.

Definition 1.4. We define

$$gCC = \{\mathbf{x}^{\mathbf{m}} \cdot \phi_Z : \mathbf{m} \in \mathbb{N}_0^r, Z \in \mathsf{Irr}^{\mathrm{sr}}(\Lambda), \text{ and } \mathbf{m} \cdot \mathbf{dim}(Z) = 0\}$$

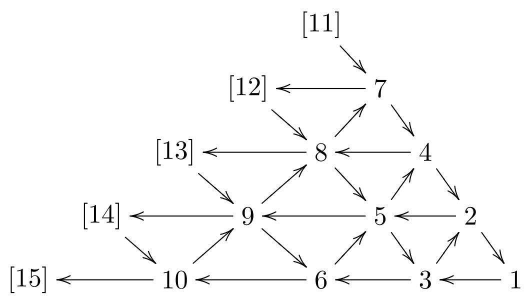
the set of *generic CC-functions*.

Remark 1.5.

1. The cluster monomials are contained in gCC.
2. The map $G : gCC \to \mathbb{Z}^r$ given by $\mathbf{x}^m \phi_Z \mapsto \mathbf{m} + g_Z$ is an isomorphism (requires our set up; in general just injective) (Plamondon).
3. The set gCC is independent of initial seed (Plamondon).

Theorem 1.6 (GLS)**.** For unipotent cells $N^w := N \cap B_- w B_-$ (where B is a maximal unipotent group, B_- is Borel, and w is an element of the Weyl group) in a Kac-Moody group with symmetric Cartan matrix, we have $\mathbb{C}[N^w] = A(\widetilde{Q}_{\mathbf{i}}, \mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{C}$ for some ice quiver $\widetilde{Q}$ and reduced expression $\mathbf{i}$ of w. Moreover, gCC is the dual semi-canonical basis (in particular it is a basis).

Example 1.7. For $G = \mathrm{SL}_{n+1}(\mathbb{C})$ (type A_n), N^{w_0} is the upper triangular matrices with 1's on the main diagonal. Then $\widetilde{Q}$ is



and $\mathbf{i} = 123451234123121$.

2 Generic Bases for Surfaces

Study surfaces S with $\partial S \neq \varnothing$ and without punctures (in the interior). From a triangulation τ by [FST] we get a quiver Q and by [Lambardini] we get a potential W. In this setup $\Lambda = \mathbb{C}Q/(\partial W)$ is a *gentle algebra* (Butler-Ringel).

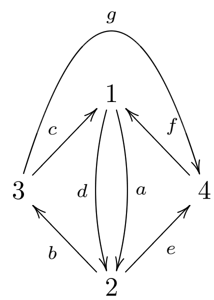
Definition 2.1. A basis algebra $\mathbb{C}Q/I$ is *gentle* if:

1. No more than two arrows leave any given vertex and no more than two arrows enter any vertex (2 in and 2 out is ok).
2. I is generated by paths of length 2.
3. If a vertex has two arrows exiting and one entering, then at most one of the two compositions is in I (and dually).

A gentle algebra is a *surface algebra* if also we have that if a sequence of two arrows is in I then they lie on a 3-cycle so that all of the compositions of two arrows are in I.

For such algebras, the indecomposable modules are classified by *string* and *band* modules.

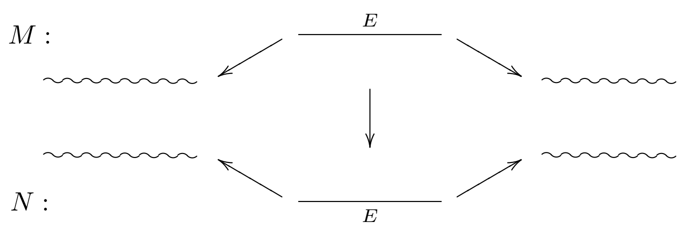
Example 2.2. The quiver Q below (modulo relations) is a surface algebra (given by a triangulation of a torus with one boundary component and one marked point on the boundary):



A string module: $M_w:\ 3 \xrightarrow{g} 4 \xrightarrow{f} 1 \xrightarrow{a} 2 \xleftarrow{d} 1 \xrightarrow{a} 2 \xleftarrow{d} 1$ with $\mathbf{dim} M_w = 3211$.

A band module: $N_{\nu,\lambda,n}:\ \mathbb{C}^n \xrightarrow{c=id} \mathbb{C}^n \xleftarrow{f=id} \mathbb{C}^n \xleftarrow{g=\partial_\nu(\lambda)} \mathbb{C}^n$.

With this description, homomorphisms between string modules become easy to compute: The evident morphism



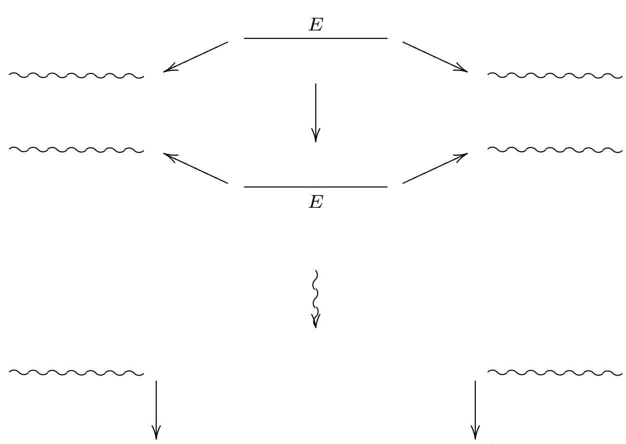
gives a basis for $\mathrm{Hom}_\Lambda(M, N)$.

2.1 Linear Independence of gCC

Lemma 2.3. For each surface S with $\partial S \neq \varnothing$, there is a triangulation τ such that the exchange matrix $B = B(\tau)$ has the following property: If B has rank t, then there are t linearly independent columns such that all other columns are *non-negative* combinations of them.

2.2 Algebraic Skein Relations

Homomorphisms $M \to \tau M$ (not radical) induce *skein relations*:



Using these skein relations and the previous lemma we can show that gCC spans the cluster algebra and contains the cluster algebra.