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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SLHERMES@BRANDEIS.EDU

Speaker's Name: P. Pylyavskyy

Talk Title: Total positivity, Loop Groups, and Electrical Networks

Date: 10 / 1 / 12 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Totally non-negative, Loop groups, canonical bases, solitonic systems, electrical networks

Please summarize the lecture in 5 or fewer sentances: The speaker introduced the notion of total non-negativity for matrices and showed they are classified by certain canonical matrices. He did the same for formal power series, and extended these results to loop groups. Connections to electrical networks were mentioned.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
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- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Total Positivity, Loop Groups, and Electrical Networks

P. Pylyavskyy November 1, 2012

Joint with Thomas Lam.

1 Total Non-Negativity

Definition 1.1. A matrix is *totally non-negative* if all minors are non-negative.

In $N \subset GL_n$ (upper triangular matrices), the totally non-negative part $N_{\geq 0}$ is a semigroup.

Theorem 1.2 (Lowner-Whitney)**.** Any element of $N_{\geq 0}$ factors into exponents of *Chevalley generators* $e_i(a)$ with non-negative parameters (e.g. the Chevalley generator $e_2(a)$ is the matrix with 1's on the main diagonal, and everything else 0 except for the 2nd entry in the diagonal one above the main one).

Theorem 1.3 (Lustig)**.** Let $s_{i_1} \cdots s_{i_\ell} = w$ be a reduced expression of w. Then the map $\mathbb{R}^\ell_{>0} \to N_{\geq 0} \cap B_- w B_-$ given by $(a_1, \ldots, a_\ell) \mapsto e_{i_1}(a_1) \cdots e_{i_\ell}(a_\ell)$ is a bijection.

What if we choose a different reduced expression? Then $e_i(a)e_j(b) = e_j(b)e_i(a)$ for $|i-j| > 1$ and

$$e_i(a)e_j(b)e_i(c) = e_j\left(\frac{bc}{a+c}\right) e_i(a+c) e_j\left(\frac{ab}{a+c}\right)$$

for $|i-j| = 1$.

In $\mathbb{R}((t))$, consider power series of the form $a(t) = 1 + a_1 t + a_2 t^2 + \ldots$. We want to unfold this into a(n infinite) matrix:

$$\begin{pmatrix} \ddots & & & & \\ & 1 & a_1 & a_2 & \\ & & 1 & a_1 & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$

Definition 1.4. We say that $a(t)$ is *totally non-negative* if this matrix is totally non-negative. Such power series form a semi-group as before.

Examples 1.5.

1. $a(t) = 1 + \alpha t$ for $\alpha \geq 0$ is totally non-negative.
2. A product of such $a(t)$ will again be totally non-negative.
3. $1/(1-\beta t)$ for $\beta \geq 0$ is totally non-negative:

$$\frac{1}{1-\beta t} \rightsquigarrow \begin{pmatrix} \ddots & & & & & \\ & 1 & \beta & \beta^2 & \beta^3 & \\ & & 1 & \beta & \beta^2 & \\ & & & 1 & \beta & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}$$

4. A product of such is again totally non-negative.
5. The function $e^\gamma t$ for $\gamma \geq 0$ is totally non-negative:

$$e^{\gamma t} \rightsquigarrow \begin{pmatrix} \ddots & & & & & \\ & 1 & \gamma & \gamma^2/2! & \gamma^3/3! & \\ & & 1 & \gamma & \gamma^2/2! & \\ & & & 1 & \gamma & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}$$

Theorem 1.6 (Edrei-Thomas)**.** Any totally non-negative function $a(t)$ as above has form

$$a(t) = e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, $\gamma \geq 0$ and $\gamma + \sum(\alpha_i + \beta_i) < \infty$.

Theorem 1.7 (Thomas, Vershik-Kerov)**.** The following sets are in canonical bijection:

1. Normalized (i.e. $a_0 = a_1 = 1$) totally non-negative functions
2. Extremal characters of the infinite symmetric group S_∞
3. Normalized functions $Sym \to \mathbb{R}$ (the symmetric algebra) taking non-negative values on Schur functions
4. extremal Markov chains on Young's lattice.

Theorem 1.8 (Vershik-Kerov)**.** Recall $S_\infty = (S_1 \subset S_2 \subset S_3 \subset \cdots)$. Chose a sequence of Young diagrams $\lambda_1 \subset \lambda_2 \subset \cdots$. Then the following are equivalent:

1. The limit $\displaystyle\lim_{n\to\infty} \frac{\chi_{\lambda_n}(w)}{f^{\lambda_n}}$ exists for all w
2. The limit $\displaystyle\lim_{n\to\infty} \frac{(i\text{-th row (column) of } \lambda_n)}{n}$ exists.

So the sequence $\lambda_1 \subset \lambda_2 \subset \cdots$ limits to a character of S_∞ iff the two limits exist. Such a character is called an *extremal character*.

2 Loop Groups

Definition 2.1. The *formal loop group* $GL_n(\mathbb{R}((t)))$ consists of invertible $n \times n$ matrices whose entries are formal Laurent series in t.

Example 2.2.

$$\begin{pmatrix} 1+t^2 & 2+5t \\ -1-t & -4t^2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \ddots & & \ddots & & \ddots & & & \\ & 1 & 2 & 0 & 5 & 1 & 0 & \\ & -1 & 1 & -1 & 0 & 0 & -4 & \\ & & & 1 & 2 & 0 & 5 & \ddots \\ & & & -1 & 1 & -1 & 0 & \\ & & & & & 1 & 2 & \ddots \\ & & & & & -1 & 0 & \\ & & & & & & & \ddots \end{pmatrix}$$

(the 2×2 blocks repeat along the diagonals; the first set of blocks is the constant terms, the second the t therms, the third the t^2 terms, etc.).

Generators: Analogues of the generators from before:

1. Whirls $M(x_1, \ldots, x_n)$:

$$\begin{pmatrix} \ddots & \ddots & & & & & & & \\ & 1 & x_1 & & & & & & \\ & & 1 & x_2 & & & & & \\ & & & 1 & \ddots & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & 1 & x_1 & & \\ & & & & & & 1 & x_2 & \\ & & & & & & & 1 & \ddots \\ & & & & & & & & \ddots \end{pmatrix}$$

2. Curls $N(x_1, \ldots, x_n)$:

$$\begin{pmatrix} \ddots & \ddots & & & & & & & \\ & 1 & x_1 & x_1 x_2 & & & & & \\ & & 1 & x_2 & & & & & \\ & & & 1 & \ddots & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & 1 & x_1 & x_1 x_2 & \\ & & & & & & 1 & x_2 & \\ & & & & & & & 1 & \ddots \\ & & & & & & & & \ddots \end{pmatrix}$$

3. Chevalley generators $e_i(a)$:

$$\begin{pmatrix} \ddots & \ddots & & & & & & & \\ & 1 & a & & & & & & \\ & & 1 & 0 & & & & & \\ & & & 1 & \ddots & & & & \\ & & & & \ddots & 0 & & & \\ & & & & & 1 & a & & \\ & & & & & & 1 & 0 & \\ & & & & & & & 1 & \ddots \\ & & & & & & & & \ddots \end{pmatrix}$$

Theorem 2.3. Any $X \in N_{\geq 0}$ = upper triangular part of $GL_n(\mathbb{R}((t)))$ can be written uniquely as

$$X = \left(\prod_{i=1}^{\infty} M_i \right) Y \left(\prod_{i=-\infty}^{-1} N_i \right)$$

where M_i, N_i have non-negative parameters, and Y is doubly entire and totally non-negative.

Why Interesting:

1. Relation to canonical bases
2. Relation with discrete solitonic systems (box ball systems)
3. Infinite reduced words

Example 2.4. $n = 3$ take $e_2(\ )e_1(\ )e_2(\ )e_3(\ )e_1(\ )\dots$ which corresponds to $2123123123\dots$. After applying some braid moves get $123123123\dots$